Geometry of Surfaces III 2011

Assignment 1.

Please hand up solutions to the starred questions for marking either in the lecture on Tuesday 9th August or in the Hand-In Box on Level 6 by 5.00 pm on that same day.
The Hand-In Boxes are straight in front of you as you come out of the lifts on Level 6.

1. If $x, y, \in \mathbb{R}^n$ show that

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad \textbf{Cauchy's inequality},$$

and

$$\|x + y\| \leq \|x\| + \|y\| \qquad \textbf{Triangle inequality}.$$

2*. If $x, y, \in \mathbb{R}^n$ show that

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Hint: If you are not sure what to do about the modulus sign try replacing this single inequality by a pair of inequalities which are equivalent to it.

3*. Using (2) or otherwise show that if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \|x_n\| = \|x\|$.

4. Let (x_n) and (y_n) be sequences in $\mathbb{R}^n$. Assuming that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ show that $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle$. Hint: you need to consider $|\langle x_n, y_n\rangle - \langle x, y\rangle|$ and make a cunning insertion of some $\langle z, w\rangle - \langle z, w\rangle$ so you can apply Cauchy's inequality. You might need (3) as well.

5*. Let $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Show that if $\|x\|^2 = \sum_{i=1}^n |x^i|^2$ then

$$\max\{|x^1|, \ldots, |x^n|\} \leq \|x\| \leq \sqrt{n}\max\{|x^1|, \ldots, |x^n|\}.$$

6. Let $\epsilon > 0$ and $x \in \mathbb{R}^n$. Show that the open ball $B(x, \epsilon)$ is an open set in $\mathbb{R}^n$.

7. For this question you only need to draw and count things. I'm not expecting precise proofs. We say a surface has been triangulated if you have drawn a collection of triangles on it covering it completely and overlapping only along edges and vertices. For any triangulation $\mathcal{T}$ let f be the number of faces, e the number of edges and v the number of vertices. We define the *Euler class* χ of the triangulation by

$$\chi(\Sigma, \mathcal{T}) = f - e + v.$$

It is a remarkable fact that the Euler class depends only on the surface, not on the triangulation.

(a) Calculate the Euler class of the sphere by calculating it for a tetrahedron or triangulating a cube.

(b) Calculate the Euler class of a torus by thinking of it as a square with appropropriate edges and vertices identified. You can just divide the square into two triangles along a diagonal.

(c) Calculate the Euler class of a surface of genus 2.

(d) Can you make a conjecture for the Euler class of a surface of genus g? Can you prove it by chopping up a surface of genus g into triangles?