2009 Exam

1. Give a True (T) or False (F) answer to each of the following statements. In each case give a *short* reason for your answer.
   (i) Every group of order 13 is cyclic.
   (ii) Any two elements of S_5 of the same order are conjugate.
   (iii) C_{15} has two composition series.
   (iv) If $|G| = 36$ and $|H| = 7$ then any homomorphism $f\colon G \to H$ has $\ker(f) = G$.
   (v) $C_3 \times C_9 \cong C_{27}$.
   (vi) A finite group can have 100 Sylow-5-subgroups.
   (vii) $\mathbb{Z}_6$ has zero-divisors.
   (viii) A polynomial of degree 3 in $\mathbb{Z}_7[x]$ has at most three zeros.

[24 marks]

2. Let G be a group.
   (a) Define the *center* of G and the *conjugacy class* $[x]$ of an element $x \in G$.
   (b) Show that $x \in Z(G)$ if and only if $|[x]| = 1$.
   (c) Show that if $G/Z(G)$ is a cyclic group then G is abelian.
   (d) Let G have order pq, where p and q are distinct primes.
   Use part (c) of this question to show that if G is not abelian then $Z(G)$ is trivial.
   To what group is G isomorphic if G is abelian ?

[13 marks]

3. (a) Find the torsion invariants and the rank of the abelian group

$$G = \langle A, B, C \mid a^3b^9c^{-3} = b^{12}c^{-6} = a^9b^3c^3 = e\rangle.$$

   (b) Determine all abelian groups of order 300, giving the prime power decomposition and torsion invariants for each group.

[10 marks]

2009 Supp Exam

1. State whether each of the following claims is true or false. In each case give a *short reason* to justify your answer.
   (a) The alternating group A_6 contains no element of order 6.
   (b) The groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic. (Here $\mathbb{R}^+$ denotes the set of positive real numbers.)
   (c) The group $C_2 \times C_8$ contains 8 elements of order 8.
   (d) The groups $C_2 \times C_9$ and C_{18} are isomorphic.
   (e) The ring $M_2(\mathbb{R})$ has no zero divisors.
   (f) A polynomial of degree 4 in $\mathbb{Z}_{13}[x]$ has at most four zeros.
   (g) If I is an integral domain then the ring $I[x]$ of polynomials is isomorphic to the ring $\mathcal{P}(I)$ of polynomial functions over I.
   (h) In an integral domain I, every prime is irreducible.

[24 marks]

2. (a) Find the torsion invariants and the rank of the abelian group

$$G = \langle a, b, c \mid a^3b^2 = a^9b^6c^{12} = bc^6 = e \rangle$$

   (b) Determine all abelian groups of order 675, giving the prime power decomposition and torsion invariants for each group.

[12 marks]

3. (a) Let G be a finite group and $x \in G$. Define the *conjugacy class* $[x]$ and the centralizer $C_G(x)$ of x. Show directly (i.e., without using the Orbit-Stabilizer Theorem,) that

$$|[x]| = (G : C_G(x)).$$

   (b) Consider the groups S_4 and A_4 and let $x = (1\,2\,3)$, which is an element of both groups.
      (i) Find the conjugacy class and the centralizer of x in S_4.
      (ii) Find the centralizer of x in A_4. Hence find the number of elements in the conjugacy class of x in A_4.

[13 marks]