1. Give a True (T) or False (F) answer to each of the following statements. In each case give a short reason for your answer.
   (i) Every group of order 13 is cyclic.
   (ii) Any two elements of $S_5$ of the same order are conjugate.
   (iii) $C_{15}$ has two composition series.
   (iv) If $|G| = 36$ and $|H| = 7$ then any homomorphism $f : G \rightarrow H$ has $\ker(f) = G$.
   (v) $C_3 \times C_9 \cong C_{27}$.
   (vi) A finite group can have 100 Sylow 5-subgroups.
   (vii) $\mathbb{Z}_6$ has zero-divisors.
   (viii) A polynomial of degree 3 in $\mathbb{Z}_7[x]$ has at most three zeros.  [24 marks]

2. Let $G$ be a group.
   (a) Define the center of $G$ and the conjugacy class $[x]$ of an element $x \in G$.
   (b) Show that $x \in Z(G)$ if and only if $|[x]| = 1$.
   (c) Show that if $G/Z(G)$ is a cyclic group then $G$ is abelian.
   (d) Let $G$ have order $pq$, where $p$ and $q$ are distinct primes.
       Use part (c) of this question to show that if $G$ is not abelian then $Z(G)$ is trivial.
       To what group is $G$ isomorphic if $G$ is abelian?  [13 marks]

3. (a) Find the torsion invariants and the rank of the abelian group
   $$G = \langle A, B, C | a^3b^3c^{-3} = b^{12}c^{-6} = a^9b^3c^3 = e \rangle.$$  
   (b) Determine all abelian groups of order 300, giving the prime power decomposition and torsion invariants for each group.  [10 marks]
1. State whether each of the following claims is true or false. In each case give a short reason to justify your answer.

(a) The alternating group $A_6$ contains no element of order 6.

(b) The groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic. (Here $\mathbb{R}^+$ denotes the set of positive real numbers.)

(c) The group $C_2 \times C_5$ contains 8 elements of order 8.

(d) The groups $C_2 \times C_9$ and $C_{18}$ are isomorphic.

(e) The ring $M_2(\mathbb{R})$ has no zero divisors.

(f) A polynomial of degree 4 in $\mathbb{Z}_{13}[x]$ has at most four zeros.

(g) If $I$ is an integral domain then the ring $I[x]$ of polynomials is isomorphic to the ring $\mathcal{P}(I)$ of polynomial functions over $I$.

(h) In an integral domain $I$, every prime is irreducible.

[24 marks]

2. (a) Find the torsion invariants and the rank of the abelian group

$$G = \langle a, b, c \mid a^3b^2 = a^9b^6c^{12} = b^6c^6 = 1 \rangle$$

(b) Determine all abelian groups of order 675, giving the prime power decomposition and torsion invariants for each group.

[12 marks]

3. (a) Let $G$ be a finite group and $x \in G$. Define the conjugacy class $[x]$ and the centralizer $C_G(x)$ of $x$. Show directly (i.e., without using the Orbit-Stabilizer Theorem,) that

$$||x|| = (G : C_G(x))$$

(b) Consider the groups $S_4$ and $A_4$ and let $x = (1 2 3)$, which is an element of both groups.

(i) Find the conjugacy class and the centralizer of $x$ in $S_4$.

(ii) Find the centralizer of $x$ in $A_4$. Hence find the number of elements in the conjugacy class of $x$ in $A_4$.

[13 marks]