Examination in School of Mathematical Sciences
Semester 1, 2009

004094 Groups and Rings III
PURE MTH 3007

Official Reading Time: 10 mins
Writing Time: 180 mins
Total Duration: 190 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 100

Instructions

• Attempt all questions.
• Begin each answer on a new page.
• Examination materials must not be removed from the examination room.

Materials

• 1 Blue book is provided.
• Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.
1. Give a True (T) or False (F) answer to each of the following statements. In each case give a short reason for your answer.

(i) Every group of order 13 is cyclic.
(ii) Any two elements of $S_5$ of the same order are conjugate.
(iii) $C_{15}$ has two composition series.
(iv) If $|G| = 36$ and $|H| = 7$ then any homomorphism $f : G \to H$ has $\ker(f) = G$.
(v) $C_3 \times C_9 \cong C_{27}$.
(vi) A finite group can have 100 Sylow 5-subgroups.
(vii) $\mathbb{Z}_6$ has zero-divisors.
(viii) A polynomial of degree 3 in $\mathbb{Z}_7[x]$ has at most three zeros.

[24 marks]

2. Let $G$ be a group.

(a) Define the center of $G$ and the conjugacy class $[x]$ of an element $x \in G$.
(b) Show that $x \in Z(G)$ if and only if $|[x]| = 1$.
(c) Show that if $G/Z(G)$ is a cyclic group then $G$ is abelian.
(d) Let $G$ have order $pq$, where $p$ and $q$ are distinct primes.
   Use part (c) of this question to show that if $G$ is not abelian then $Z(G)$ is trivial.
   To what group is $G$ isomorphic if $G$ is abelian?

[13 marks]

3. (a) Find the torsion invariants and the rank of the abelian group

\[ G = \langle A, B, C \mid a^3 b^9 c^{-3} = b^{12} c^{-6} = a^9 b^3 c^3 = e \rangle. \]

(b) Determine all abelian groups of order 300, giving the prime power decomposition and torsion invariants for each group.

[10 marks]

Please turn over for page 3
4. (a) Let $G$ be a finite group acting on a set $X$.
   (i) Define what is meant by the *orbit* and *stabilizer subgroup* of an element.
   (ii) State the Orbit-Stabilizer Theorem.
   (iii) Let $r$ be the number of orbits of $G$ on $X$ and for each $g \in G$ let
   $\quad X_g = \{ x \in X \mid gx = x \}$.
   State Burnside’s Theorem which gives a formula for $r$ in terms of the $X_g$.

   (b) We wish to paint each edge of a triangle with one of $n$ different coloured paints. We are allowed to paint adjacent edges with the same coloured paint. Two paintings are considered to be the same if we can act by a symmetry of the triangle (i.e an element of $S_3$) until they look the same. Use Burnside’s Theorem to give an expression for the number of different paintings.

   [13 marks]

5. (a) Define a Sylow $p$-subgroup of a finite group $G$.
   (b) State Sylow’s three theorems on the existence, conjugacy and number of Sylow $p$-subgroups of a finite group $G$.
   (c) If $G$ has exactly one Sylow $p$-subgroup $P$ explain why $P$ is normal in $G$.
   (d) Assume that $G$ has order 77 and show that $G \cong C_{77}$.

   [14 marks]

6. (a) Define what it means for a ring $R$ to be:
   (i) an *integral domain*
   (ii) a *field*.
   If you need them define the notions of *unit* and *zero-divisor*.
   (b) Give examples of a commutative ring with identity which is not an integral domain and an integral domain which is not a field.
   (c) Give the definition of an *ideal* and a *maximal ideal* of a ring $R$.
   (d) Show that the only ideals in a field $F$ are 0 and $F$.

   [13 marks]
7. Consider the integral domain \( \mathbb{Z}(\sqrt{-3}) \) with norm
\[ N(a + b\sqrt{-3}) = a^2 + 3b^2. \]
You may use the fact that \( N(\alpha \beta) = N(\alpha)N(\beta) \) in this question.
(a) Prove that if \( \alpha \in \mathbb{Z}(\sqrt{-3}) \) is a unit then \( N(\alpha) = 1 \). Hence find all units of \( \mathbb{Z}(\sqrt{-3}) \).
(b) Show that if \( N(c) = p \), \( p \) a prime in \( \mathbb{Z} \), then \( c \) is an irreducible element of \( \mathbb{Z}(\sqrt{-3}) \).
(c) Show that 2 is irreducible in \( \mathbb{Z}(\sqrt{-3}) \).
(d) Show that 2 is not a prime in \( \mathbb{Z}(\sqrt{-3}) \) by considering \((1 + \sqrt{-3})(1 - \sqrt{-3})\).

[13 marks]