THE UNIVERSITY OF ADELAIDE AUSTRALIA

Examination in School of Mathematical Sciences

Semester 1, 2009

004094	**Groups and Rings III**
	PURE MTH 3007

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 100

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Give a True (T) or False (F) answer to each of the following statements. In each case give a *short* reason for your answer.
   (i) Every group of order 13 is cyclic.
   (ii) Any two elements of S_5 of the same order are conjugate.
   (iii) C_{15} has two composition series.
   (iv) If $|G| = 36$ and $|H| = 7$ then any homomorphism $f\colon G \to H$ has $\ker(f) = G$.
   (v) $C_3 \times C_9 \cong C_{27}$.
   (vi) A finite group can have 100 Sylow 5-subgroups.
   (vii) $\mathbb{Z}_6$ has zero-divisors.
   (viii) A polynomial of degree 3 in $\mathbb{Z}_7[x]$ has at most three zeros.

[24 marks]

2. Let G be a group.
   (a) Define the *center* of G and the *conjugacy class* $[x]$ of an element $x \in G$.
   (b) Show that $x \in Z(G)$ if and only if $|[x]| = 1$.
   (c) Show that if $G/Z(G)$ is a cyclic group then G is abelian.
   (d) Let G have order pq, where p and q are distinct primes.
   Use part (c) of this question to show that if G is not abelian then $Z(G)$ is trivial.
   To what group is G isomorphic if G is abelian ?

[13 marks]

3. (a) Find the torsion invariants and the rank of the abelian group

$$G = \langle A, B, C \mid a^3b^9c^{-3} = b^{12}c^{-6} = a^9b^3c^3 = e\rangle.$$

   (b) Determine all abelian groups of order 300, giving the prime power decomposition and torsion invariants for each group.

[10 marks]

Please turn over for page 3

4. (a) Let G be a finite group acting on a set X.
   (i) Define what is meant by the *orbit* and *stabilizer subgroup* of an element.
   (ii) State the Orbit-Stabilizer Theorem.
   (iii) Let r be the number of orbits of G on X and for each $g \in G$ let

$$X_g = \{x \in X \mid gx = x\}.$$

   State Burnside's Theorem which gives a formula for r in terms of the X_g.

   (b) We wish to paint each edge of a triangle with one of n different coloured paints. We are allowed to paint adjacent edges with the same coloured paint. Two paintings are considered to be the same if we can act by a symmetry of the triangle (i.e an element of S_3) until they look the same. Use Burnside's Theorem to give an expression for the number of different paintings.

[13 marks]

5. (a) Define a Sylow p-subgroup of a finite group G.
   (b) State Sylow's three theorems on the existence, conjugacy and number of Sylow p-subgroups of a finite group G.
   (c) If G has exactly one Sylow p-subgroup P explain why P is normal in G.
   (d) Assume that G has order 77 and show that $G \simeq C_{77}$.

[14 marks]

6. (a) Define what it means for a ring R to be:
   (i) an *integral domain*
   (ii) a *field*.

   If you need them define the notions of *unit* and *zero-divisor*.
   (b) Give examples of a commutative ring with identity which is not an integral domain and an integral domain which is not a field.
   (c) Give the definition of an *ideal* and a *maximal ideal* of a ring R.
   (d) Show that the only ideals in a field F are 0 and F.

[13 marks]

7. Consider the integral domain $\mathbb{Z}(\sqrt{-3})$ with norm

$$N(a + b\sqrt{-3}) = a^2 + 3b^2\,.$$

You may use the fact that $N(\alpha\beta) = N(\alpha)N(\beta)$ in this question.

(a) Prove that if $\alpha \in \mathbb{Z}(\sqrt{-3})$ is a unit then $N(\alpha) = 1$. Hence find all units of $\mathbb{Z}(\sqrt{-3})$.

(b) Show that if $N(c) = p$, p a prime in $\mathbb{Z}$, then c is an irreducible element of $\mathbb{Z}(\sqrt{-3})$.

(c) Show that 2 is irreducible in $\mathbb{Z}(\sqrt{-3})$.

(d) Show that 2 is not a prime in $\mathbb{Z}(\sqrt{-3})$ by considering $(1 + \sqrt{-3})(1 - \sqrt{-3})$.

[13 marks]