NB: These are brief comments NOT examples of the kinds of solutions I expect you to write!

Q1. (i) T, (ii) F (iii) T (iv) T (v) F (vi) F (vii) T (viii) T

Q2. (a) Lectures (b) Lectures (c) Lectures (d) From Lagrange $|Z(G)| = 1$, $p$, $q$ or $pq$. As $G$ is not abelian $|Z(G)| \neq pq$. If $|Z(G)| = p$ then $|G/Z(G)| = q$ so $G/Z(G)$ is cyclic so $G$ is abelian a contradiction. Similar if $|Z(G)| = q$ so $|Z(G)| = 1$. If $G$ is abelian then it is finitely generated and abelian so isomorphic to $C_p \times C_q \cong C_{pq}$.

Q3. (a) Torsion invariants 3 and 6, rank is 1. (b) I'll list the torsion invariants you can deduce the decomposition into cyclic groups. 10, 30, 5, 60, 2, 150, 300.

Q4. (a) Lectures (b) CE 4, Q 5

Q5 (a) Lectures (b) Lectures (c) Lectures. (d) $|G| = 7 \times 11$ and 11 is not congruent to 1 modulo 7 so proceed as in lectures.

Q6. (a) Lectures (b) $\mathbb{Z}_6$ and $\mathbb{Z}$. (c) Lectures. (d) Lectures

Q7. (a) Lectures and then note that for a unit we must have $a^2 + 3b^2 = 1$ whose only solutions are $a = \pm 1$ with $b = 0$. So only units are $\pm 1$. (b) Lectures. (c) $N(2) = 4$. If $2 = xy$ then $4 = N(2) = N(x)N(y)$. As we don’t want either of $x$ or $y$ to be a unit must have $N(x) = N(y) = 2$. But $a^2 + 3b^2 = 2$ is not possible to solve for $a$ and $b$ integers. So 2 is irreducible. (d) We have $(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4$ is divisible by 2. However 2 does not divide either factor as the result would be $1/2 \pm 1/2\sqrt{-3} \notin \mathbb{Z}(\sqrt{-3})$.

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2010/6/4