Groups and Rings III 2010

Tutorial Exercise 6.

Please try before the tutorial on Tuesday 1st June.

1. Find all zeros, and hence factorise the following polynomials:

(a) $f(x) = x^3 - x^2 + 2x - 2$ in $\mathbb{Z}_3[x]$;

(b) $g(x) = x^4 - 4x^3 + x^2 - 4x$ in $\mathbb{Z}_5[x]$;

(c) $h(x) = x^3 + 2x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$;

2. Let F be a field and $a \in F$.

(a) Show that $\phi\colon F[x] \to F$ defined by $\phi(f(x)) = \bar{f}(a)$ is a ring homomorphism.

(b) Use the Factor Theorem to show that $\ker(\phi) = \langle x - a \rangle$.

(c) Deduce that $F[x]/\langle x - a \rangle \cong F$.

3. Let $\{0\} \neq I \subset \mathbb{Z}$ be an ideal.

(a) If n is any number in I show that $\langle n \rangle \subset I$.

(b) Show that I contains positive numbers.

(c) If n is the smallest positive number in I show that $\langle n \rangle = I$ using the Division Algorithm for integers.

(d) Conclude that every ideal in $\mathbb{Z}$ is a principal ideal.

4. Let D be an integral domain. Show that

(a) $b = ar$ for some $r \in D$ if and only if $\langle a \rangle \supseteq \langle b \rangle$

(b) $\langle a \rangle = D$ if and only if a is a unit

(c) $\langle a \rangle = \langle b \rangle$ if and only if a and b are associates.

5. Let a and b be positive integers and let $\langle a, b \rangle$ be the ideal generated by a and b. Using the fact that every ideal in $\mathbb{Z}$ is principal show that

$$\langle a, b \rangle = \langle \gcd(a, b) \rangle.$$

Deduce that there exist integers m and n such that $ma + nb = \gcd(a, b)$ and if c is a common divisor of a and b then $c \mid \gcd(a, b)$.