Groups and Rings III 2010

Tutorial Exercise 3.

Please try before the tutorial on Tuesday 30th March.

1. (a) If $H < G$ prove that $N_G(H)$ is a subgroup of G containing H.

(b) In S_3 find all the conjugates of $H = \{1, (12)\}$.

(c) Find the normaliser of H in S_3. You can use results from class where we listed the left and right cosets of H. Verify the formula for the number of conjugacy classes we proved in class.

2. If H and K are subgroups of G let $HK = \{hk \mid h \in H, k \in K\}$.

(a) If H and K are subgroups of G with $H \triangleleft G$ show that HK is a subgroup of G.

(b) Let H and K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$.

3. Let G be a group with $N \triangleleft G$ and $N \neq \langle e \rangle$.

(a) Prove that N is a union of conjugacy classes in G.

(b) If G is a p-group show that $Z(G) \cap N \neq \langle e \rangle$. (Hint: Use the same idea that was used in class to show that $Z(G) \neq \langle e \rangle$ for p-groups.)

4. Let $f : G \to H$ be a homomorphism of groups. If $K \subset G$ define $f(K) = \{f(k) \mid k \in K\} \subset H$ and if $L \subset H$ define $f^{-1}(L) = \{g \in G \mid f(g) \in L\} \subset G$.

(a) If $K < G$ show that $f(K) < H$.

(b) If $L < H$ show that $f^{-1}(L) < G$.

(c) If $K \triangleleft G$ and f is onto show that $f(K) \triangleleft H$.

(d) If $L \triangleleft H$ show that $f^{-1}(L) \triangleleft G$.

5. For $a \neq 0$ and b in $\mathbb{Z}_7$ define a function

$$F_{a,b} : \mathbb{Z}_7 \to \mathbb{Z}_7$$

by $F_{a,b}(x) = ax + b$. The set of all these functions forms a group G under composition. (You may assume this.)

(a) Determine $|G|$ and calculate $F_{a,b} \circ F_{c,d}(x) = F_{ab}(F_{c,d}(x))$. Use this to find e, f such that $F_{a,b} \circ F_{c,d} = F_{e,f}$.

(b) Show that G is non-abelian.

(c) Show that the mapping $f : G \to \mathbb{Z}_7^\times$ given by $f(F_{a,b}) = a$ is a homomorphism. Find $\ker(f)$. Write down a normal subgroup of G.

6. (a) Let C_{12} be generated by x and let $H = \langle x^4 \rangle$. List the elements of the factor group C_{12}/H. Find a generator of C_{12}/H and hence show that it is cyclic.

(b) Find all the composition series of C_{24}. Show that in each case the length of the composition series and the set of quotients is the same.