Groups and Rings III 2010

Tutorial Exercise 2.

Please try before the tutorial on Tuesday 16th March.

1. Let
\[ D = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}_5^\times, \alpha \beta \neq 0 \right\}; \]
i.e. \( D \) is the subgroup of diagonal matrices of \( GL(2, \mathbb{Z}_5^\times) \).

(a) Show that \( D \) is a subgroup.

(b) Calculate \( |D| \) and show that \( D \) is abelian.

(c) Is \( D \) cyclic? If \( D \) is cyclic find a generator; if not, find the smallest number of matrices \( A, B, C, \ldots \) such that \( D = \langle A, B, C, \ldots \rangle \).

(d) Determine \( |D \cap SL(2, \mathbb{Z}_5^\times)| \).

2. If \( c = (i_1, i_2, \ldots, i_k) \in S_n \) find a decomposition of \( c \) into transpositions and hence show that \( c \) is even (odd) if and only if \( |c| \) is odd (even). (Hint: You can get away with \( k - 1 \) transpositions.)

3. Consider the following permutations in \( S_6 \), the symmetric group on 6 letters.
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}. \]

(a) Write each of \( \sigma \) and \( \tau \) as a product of disjoint cycles and find their orders.

(b) Write down \( \sigma^{-1} \) and find \( \sigma^{-1} \tau \).

(c) Is \( \sigma^{-1} \tau \) an even or odd permutation?

4. Recall that \( \mathbb{R}^\times = \mathbb{R} - \{0\} \) and let \( \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \subset \mathbb{R}^\times \).

(a) Show that \( (\mathbb{R}^+, \times) \) is a subgroup of \( (\mathbb{R}^\times, \times) \).

(b) Show that \( (\mathbb{R}^+, \times) \) is isomorphic to the group \( (\mathbb{R}, +) \). (Hint: Think about log and exp.)

(c) Explain why \( (\mathbb{R}, +) \) is not isomorphic to \( (\mathbb{R}^\times, \times) \). (Hint: Elements of order two.)

(d) Is \( (\mathbb{Q}, +) \) cyclic?

5. Consider the Klein four-group \( V_4 \) which is group of symmetries of a rectangle given, as a subgroup of \( S_4 \), as
\[ V_4 = \{1, (13)(24), (12)(43), (14)(32)\} \]
Write down the multiplication tables of \( V_4 \) the group of symmetries of a rectangle and \( U_4 \) the group of fourth roots of unity. Are they isomorphic? Justify your answer. (Hint: Think about orders of elements.)