# Groups and Rings III 2010

## Tutorial Exercise 2.

**Please try before the tutorial on Tuesday 16th March.**

1. Let $D = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{Z}_5^\times, \alpha\beta \neq 0 \right\}$; i.e. $D$ is the subgroup of diagonal matrices of $GL(2, \mathbb{Z}_5^\times)$.

(a) Show that $D$ is a subgroup.

(b) Calculate $|D|$ and show that $D$ is abelian.

(c) Is $D$ cyclic? If $D$ is cyclic find a generator; if not, find the smallest number of matrices $A, B, C, ...$ such that $D = \langle A, B, C, ... \rangle$.

(d) Determine $|D \cap SL(2, \mathbb{Z}_5^\times)|$.

2. If $c = (i_1, i_2, \cdots, i_k) \in S_n$ find a decomposition of $c$ into transpositions and hence show that $c$ is even (odd) if and only if $|c|$ is odd (even). (Hint: You can get away with $k - 1$ transpositions.)

3. Consider the following permutations in $S_6$, the symmetric group on 6 letters.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}.$$

(a) Write each of $\sigma$ and $\tau$ as a product of disjoint cycles and find their orders.

(b) Write down $\sigma^{-1}$ and find $\sigma^{-1}\tau$.

(c) Is $\sigma^{-1}\tau$ an even or odd permutation?

4. Recall that $\mathbb{R}^\times = \mathbb{R} - \{0\}$ and let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \subset \mathbb{R}^\times$.

(a) Show that $(\mathbb{R}^+, \times)$ is a subgroup of $(\mathbb{R}^\times, \times)$.

(b) Show that $(\mathbb{R}^+, \times)$ is isomorphic to the group $(\mathbb{R}, +)$. (Hint: Think about log and exp.)

(c) Explain why $(\mathbb{R}, +)$ is not isomorphic to $(\mathbb{R}^\times, \times)$. (Hint: Elements of order two.)

(d) Is $(\mathbb{Q}, +)$ cyclic ?

5. Consider the Klein four-group $V_4$ which is group of symmetries of a rectangle given, as a subgroup of $S_4$, as

$$V_4 = \{1, (13)(24), (12)(43), (14)(32)\}$$

Write down the multiplication tables of $V_4$ the group of symmetries of a rectangle and $U_4$ the group of fourth roots of unity. Are they isomorphic ? Justify your answer. [Hint: Think about orders of elements.]