Groups and Rings III 2010

Tutorial Exercise 1.

Please try before the tutorial on Thursday 4th March.

1. Let $a, b \in \mathbb{Z}^+$, that is a and b are positive integers and consider

$$H = \{am + bn \mid m, n \in \mathbb{Z}\}$$

(a) Show that H is a subgroup of $\mathbb{Z}$ and contains both a and b.

(b) From lectures we can deduce from (a) that H is cyclic. Let $H = \langle d \rangle$. Explain why we can assume $d > 0$ without loss of generality.

(c) Show that $d = \gcd(a, b)$, that is d is the greatest common divisor of a and b or the largest integer dividing both a and b.

(d) Deduce that for any $a, b \in \mathbb{Z}^+$ there exists $m, n \in \mathbb{Z}$ such that $\gcd(a, b) = ma + nb$.

2. Consider the group

$$U_6 = \{z \in \mathbb{C} \mid z^6 = 1\}$$

and let $\omega = \exp(\pi i/3)$.

(a) Show that U_6 is a cyclic subgroup of $\mathbb{C}^\times = \mathbb{C} - \{0\}$ generated by ω. What is the order of ω ?

(b) Write out the multiplication table of U_6.

(c) For each element $x \in U_6$ calculate the order $|x|$ and the subgroup $\langle x \rangle$.

(d) As discussed in lectures the subgroups in (c) are all the subgroups of U_6. Draw the subgroup lattice of U_6.

3. Let G be a group.

(a) Show that a non-empty subset $\emptyset \neq H \subset G$ is a subgroup if and only if for all $x, y \in H$ we have $xy^{-1} \in H$. Is $\emptyset$ a subgroup?

(b) Show that if H and K are subgroups of G then $H \cap K$ is a subgroup of G.

4. Let G be a group.

(a) Show that for any $a, b, c \in G$ we have

(i) $|a^{-1}| = |a|$ (ii) $|b^{-1}ab| = |a|$ (iii) $|ab| = |ba|$ (iv) $|abc| = |bca| = |cab|$

In each case also show that either both sides are finite or both sides are infinite.

(b) Show that if $|x| = 2$ for all $x \neq e$ in G then G is abelian. (Hint: Consider $(ab)^2$.)