Groups and Rings III 2010

Class Exercise 6.

Not for handing in.

1. Consider the ring of Gaussian Integers, $\mathbb{Z}(i) = \{a + bi \mid a, b \in \mathbb{Z}\}$ with Euclidean valuation $\delta(a + bi) = a^2 + b^2$.

(a) For $a = 1 + 2i$, $b = 3 - i$, find $q, r \in \mathbb{Z}(i)$ such that $a = bq + r$, with $\delta(r) < \delta(b)$, where δ is the Euclidean norm for $\mathbb{Z}(i)$.

(b) For each of 2 and 3 either show that they are irreducible or factorise them into products of irreducibles in $\mathbb{Z}(i)$.

2. Recall that that the set of all units G in a ring with identity is a group with operation the ring multiplication. For the ring $\mathbb{Z}(\sqrt{2})$ use this to show that the group of units is infinite. Hint: Find a unit using the fact that it has norm 1 and then show that it has infinite order in G.

3. Consider the integral domain $D = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ with norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$. You may assume that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in D$.

(a) Prove that $\alpha \in D$ is a unit if and only if $N(\alpha) = 1$.

(b) Find all units of D.

(c) Show that if $N(\alpha) = 9$, then α is irreducible.

(d) By considering the product $(2 + \sqrt{-5})(2 - \sqrt{-5})$, show that 3 is not prime in D.

(e) Is D a unique factorization domain? Justify your answer. (Hint: In case we haven't got to it by the time you do this in a UFD primes are the same things as irreducibles.)

(Exam 2008)

4. Consider the integral domain $\mathbb{Z}(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$ with the norm $N(a + b\sqrt{10}) = a^2 - 10b^2$.

(a) Use the norm to describe the units of $\mathbb{Z}(\sqrt{10})$.

(b) By considering $a^2 \mod 10$ show that for any $x \in \mathbb{Z}(\sqrt{10})$ we have that $N(x) \mod 10$ can only be $0, 1, 4, 5, 6, 9$.

(c) Using (b) show that 2, 3, $4 + \sqrt{10}$ and $4 - \sqrt{10}$ are irreducible in $\mathbb{Z}(\sqrt{10})$.

(d) Is $\mathbb{Z}(\sqrt{10})$ a unique factorization domain ?