1. Consider the ring of Gaussian Integers, \( \mathbb{Z}(i) = \{a + bi \mid a, b \in \mathbb{Z}\} \) with Euclidean valuation \( \delta(a + bi) = a^2 + b^2 \).

(a) For \( a = 1 + 2i, \ b = 3 - i \), find \( q, r \in \mathbb{Z}(i) \) such that \( a = bq + r \), with \( \delta(r) < \delta(b) \), where \( \delta \) is the Euclidean norm for \( \mathbb{Z}(i) \).

(b) For each of 2 and 3 either show that they are irreducible or factorise them into products of irreducibles in \( \mathbb{Z}(i) \).

2. Recall that that the set of all units \( G \) in a ring with identity is a group with operation the ring multiplication. For the ring \( \mathbb{Z}(\sqrt{2}) \) use this to show that the group of units is infinite. Hint: Find a unit using the fact that it has norm 1 and then show that it has infinite order in \( G \).

3. Consider the integral domain \( D = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \) with norm \( N(a + b\sqrt{-5}) = a^2 + 5b^2 \). You may assume that \( N(\alpha\beta) = N(\alpha)N(\beta) \) for all \( \alpha, \beta \in D \).

(a) Prove that \( \alpha \in D \) is a unit if and only if \( N(\alpha) = 1 \).

(b) Find all units of \( D \).

(c) Show that if \( N(\alpha) = 9 \), then \( \alpha \) is irreducible.

(d) By considering the product \( (2 + \sqrt{-5})(2 - \sqrt{-5}) \), show that 3 is not prime in \( D \).

(e) Is \( D \) a unique factorization domain? Justify your answer. (Hint: In case we haven’t got to it by the time you do this in a UFD primes are the same things as irreducibles.)

(Exam 2008)

4. Consider the integral domain \( \mathbb{Z}(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \) with the norm \( N(a + b\sqrt{10}) = a^2 - 10b^2 \).

(a) Use the norm to describe the units of \( \mathbb{Z}(\sqrt{10}) \).

(b) By considering \( a^2 \mod 10 \) show that for any \( x \in \mathbb{Z}(\sqrt{10}) \) we have that \( N(x) \mod 10 \) can only be 0, 1, 4, 5, 6, 9.

(c) Using (b) show that 2, 3, 4 + \( \sqrt{10} \) and 4 - \( \sqrt{10} \) are irreducible in \( \mathbb{Z}(\sqrt{10}) \).

(d) Is \( \mathbb{Z}(\sqrt{10}) \) a unique factorization domain?