Groups and Rings III 2010

Class Exercise 5.

Please hand up solutions in the lecture on Thursday 27th May.

1. If R is a ring with identity show that the set of all units in R is a group under multiplication.

2. Consider the ring of real quaternions:

$$\mathbb{R}(\mathbb{H}) = \{x_1 + x_2 i + x_3 j + x_4 k \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

We define the addition and multiplication by assuming everything is linear over the real numbers and using the usual rules of multiplications in the quaternion group. E.g. $(5 + 2j)(i + 3k) = 5i + 15k + 2ji + 2jk = (5 + 2)i + (15 - 2)k = 7i + 13k$ and $(1 + 3j) + (7i + 2j + k) = 1 + 7i + 5j + k$.

(a) If $x = x_1 + x_2 i + x_3 j + x_4 k$ define $\bar{x} = x_1 - x_2 i - x_3 j - x_4 k$ and show that $x\bar{x} = \|x\|^2$ where $\|x\|$ is the usual Euclidean length of a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

(b) Deduce that any non-zero $x \in \mathbb{R}(\mathbb{H})$ is a unit.

(c) Deduce that $\mathbb{R}(\mathbb{H})$ is a skew-field.

3. Consider the set $\mathbb{Q}(\sqrt{7}) = \{a + b\sqrt{7} \mid a, b \in \mathbb{Q}\} \subset \mathbb{Q}$.

(a) Show that $\mathbb{Q}(\sqrt{7})$ is a subring of $\mathbb{Q}$.

(b) Show that $\mathbb{Q}(\sqrt{7})$ is a field.

4. Complete the following table.

Ring	Commutative	Identity	Units	Zero Divisors	Field	Integral Domain
$\mathbb{Z}$	yes	1	± 1	none	no	yes
$\mathbb{Z}(i)$						
$\mathbb{Z}_4$						
$\mathbb{Z}_3$						
$\mathbb{Q}(\sqrt{7})$						
$\mathbb{R}(\mathbb{H})$						
$M_2(\mathbb{C})$						

Note:
$\mathbb{Z}(i) = \{a + bi \mid a, b \in \mathbb{Z}\}$ is the ring of Gaussian Integers, a subring of $\mathbb{C}$.
$\mathbb{Q}(\sqrt{7}) = \{a + b\sqrt{7} \mid a, b \in \mathbb{Q}\}$ is a subring of $\mathbb{R}$.
$\mathbb{R}(\mathbb{H})$ see Question 2.
You don't have to prove everything. Just fill out the table.

5. Recall the construction in lectures of the field of quotients of an integral domain D which involved the set $S = \{(a, b) \mid a, b \in D, b \neq 0\}$.

(a) Show that the relation $(a, b) \simeq (c, d)$ if $ad = bc$ is an equivalence relation on S.

(b) Show that the addition

$$[(a, b)] + [(\alpha, \beta)] = [(a\beta + b\alpha, b\beta)]$$

is well-defined.