Groups and Rings III 2010

Class Exercise 4.

Please hand up solutions in the lecture on Thursday 13th May.

1. Consider the group $U(1) = \{z \in \mathbb{C}^\times \mid |z| = 1\}$. Identify the orbits of the action of $U(1)$ on $\mathbb{C}$ defined by $(u, w) \mapsto uw$ for $u \in U(1)$ and $w \in \mathbb{C}$. Find $S_{U(1)}(0)$ and $S_{U(1)}(1 + i\sqrt{2})$.

2. Let G act on X.

(a) Show that $S_G(x)$ is a subgroup of G.

(b) Show that $S_G(gx) = gS_G(x)g^{-1}$.

3. Let a finite group G act on a finite set X. We say the action is *free* if for every $x \in X$ the only g satisfying $gx = x$ is $g = e$.

(a) Show that if the action is free then $S_G(x) = \{e\}$ for all $x \in X$.

(b) Show that if the action is free then $X_g = \emptyset$ for all $g \neq e$.

(c) Use Burnside's theorem to show that if G acts freely on X then the number of orbits is $|X|/|G|$.

4. Let $H < G$ be finite groups.

(a) Define a function $H \times G \to G$ by $(h, g) \mapsto h \star g = gh^{-1}$. Show that this is a free action of H on G.

(b) Show that the orbits of this action of H on G are the left cosets of H.

(c) Use Burnside's theorem or the previous question to deduce Lagrange's theorem that the number of left cosets of H in G is $|G|/|H|$.

5. We wish to paint each edge of a triangle with one of n different coloured paints. We are allowed to paint adjacent edges with the same coloured paint. Two paintings are considered to be the same if we can act by a symmetry of the triangle (i.e an element of S_3) until they look the same. Use Burnside's Theorem to show that the number of different paintings is

$$\frac{n^3 + 3n^2 + 2n}{6}.$$

Exam 2009