Groups and Rings III 2010

Class Exercise 3.

Please hand up solutions in the lecture on Thursday 22nd April .

1. Find the commutator subgroup of the quaternion group $\mathbb{H}$. Verify that it is normal.

2. Let H and K be groups and define a binary operation

$$H \times K \times H \times K \to H \times K$$
$$((h_1, k_1)(h_2, k_2)) \mapsto (h_1 h_2, k_1 k_2)$$

(a) Show that this binary operation makes $H \times K$ into a group.

(b) Show that $H_0 = \{(h, e) \mid h \in H\}$ is a subgroup of $H \times K$.

(c) Show that the map $\iota_H \colon H \to H \times K$ defined by $\iota_H(h) = (h, e)$ is a one-to-one homomorphism with image H_0.

(d) Show that the map $\pi_K \colon H \times K \to K$ defined by $\pi_K((h, k)) = k$ is an onto homomorphism with kernel H_0.

3. If m and n are integers denote the least common multiple of m and n by $\mathrm{lcm}(m, n)$ and the greatest common divisor of m and n by $\gcd(m, n)$. Note that $\mathrm{lcm}(m, n)\gcd(m, n) = mn$.

(a) If $(h, k) \in H \times K$ show that $|(h, k)| = \mathrm{lcm}(|h|, |k|)$.

(b) If $(m, n) \neq 1$ show that $C_m \times C_n \not\simeq C_{mn}$.

(c) If $(m, n) = 1$ show that $C_m \times C_n \simeq C_{mn}$.

4. Let $U_1 = \{z \in \mathbb{C}^\times \mid |z| = 1\} < \mathbb{C}^\times$ and $\mathbb{R}_{>0} = \{x \in \mathbb{R}^\times \mid x > 0\}$. Show that:

(a) $\mathbb{C}^\times \simeq U_1 \times \mathbb{R}_{>0}$. (Hint: polar decomposition, i.e $z = r\exp(i\theta)$)

(b) $\mathbb{R}^\times \simeq \mathbb{Z}_2 \times \mathbb{R}_{>0}$.