Groups and Rings III 2010

Class Exercise 3.

Please hand up solutions in the lecture on Thursday 22nd April.

1. Find the commutator subgroup of the quaternion group \( H \). Verify that it is normal.

2. Let \( H \) and \( K \) be groups and define a binary operation

\[
H \times K \times H \times K \to H \times K
\]

\[
((h_1,k_1)(h_2,k_2)) \to (h_1 h_2, k_1 k_2)
\]

(a) Show that this binary operation makes \( H \times K \) into a group.

(b) Show that \( H_0 = \{(h,e) \mid h \in H\} \) is a subgroup of \( H \times K \).

(c) Show that the map \( \iota_H: H \to H \times K \) defined by \( \iota_H(h) = (h,e) \) is a one-to-one homomorphism with image \( H_0 \).

(d) Show that the map \( \pi_K: H \times K \to K \) defined by \( \pi_K((h,k)) = k \) is an onto homomorphism with kernel \( H_0 \).

3. If \( m \) and \( n \) are integers denote the least common multiple of \( m \) and \( n \) by \( \text{lcm}(m,n) \) and the greatest common divisor of \( m \) and \( n \) by \( \text{gcd}(m,n) \). Note that \( \text{lcm}(m,n) \text{gcd}(m,n) = mn \).

(a) If \( (h,k) \in H \times K \) show that \( |(h,k)| = \text{lcm}(|h|,|k|) \).

(b) If \( (m,n) \neq 1 \) show that \( C_m \times C_n \not\cong C_{mn} \).

(c) If \( (m,n) = 1 \) show that \( C_m \times C_n \cong C_{mn} \).

4. Let \( U_1 = \{z \in \mathbb{C}^\times \mid |z| = 1\} \times \mathbb{R}_>0 \) and \( \mathbb{R}_\geq = \{x \in \mathbb{R}^\times \mid x > 0\} \). Show that:

(a) \( \mathbb{C}^\times \cong U_1 \times \mathbb{R}_\geq \). (Hint: polar decomposition, i.e \( z = r \exp(i\theta) \))

(b) \( \mathbb{R}_\geq \cong \mathbb{Z}_2 \times \mathbb{R}_\geq \).