Groups and Rings III 2010

Class Exercise 2.

Please hand up solutions in the lecture on Thursday 26th March.

1. (a) If G is a finite group and $g \in G$ show that $g^{|G|} = e$.

(b) Prove Fermat's Little Theorem which says that if p is a prime and $0 < a < p$ then $a^{p-1} \equiv 1 \mod p$. [Hint: Think about the group $\mathbb{Z}_p^{\times}$.]

2. Let $g \in G$ and consider the function $\mathrm{Ad}_g : G \to G$ defined by $\mathrm{Ad}_g(x) = gxg^{-1}$. Show that Ad_g is an isomorphism for all $g \in G$.

3. Let $x \in G$. Prove that $C_G(x) = \{x \in G \mid xg = gx\}$ is a subgroup of G.

4. Determine the conjugacy classes of S_4. You can use the result from Lectures relating the conjugacy class of a permutation to its cycle structure. Pick an element π in each class and determine $C_{S_4}(\pi)$.

5. (a) If G is a group show that $Z(G)$, the centre of G, is a normal subgroup of G.

(b) Find the centre of $\mathbb{H}$.

(c) Call a matrix in $GL(n, \mathbb{C})$ a *scalar* matrix if it is a (non-zero) multiple of the identity matrix. Show that $Z(GL(n, \mathbb{C}))$ is the group of scalar matrices. (Hint: Assume X is in the centre and consider the equation $EX = XE$ where E is an elementary matrix as in Mathematics I. Try different kinds of elementary matrices.)