The Lemma below was actually proved in the Friday lecture of Week 7.

**Lemma 7.2.** Let $G$ be a finite $p$-group acting on the finite set $X$. Let

$$ F = \{ x \in X \mid g \ast x = x \text{ for all } g \in G \} . $$

Then $|F| \equiv |X|$ (mod $p$).

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### Week 8 — Lecture 18 — Tuesday 5th May.

#### 7.3. Sylow’s second and third theorems.

**Theorem 7.3. (Sylow’s Second Theorem)** Let $P$ be a Sylow $p$-subgroup of the finite group $G$ of order $p^mr$, where $r$ is coprime to $p$. If $Q$ is any $p$-subgroup of $G$ (that is, $|Q|$ is a power of $p$) then $Q < gPg^{-1}$ for some $g \in G$.

In particular, all Sylow $p$-subgroups are conjugate.

**Lemma 7.4.**

(1) Let $P$ be a Sylow $p$-subgroup of $G$ and suppose $P \trianglelefteq G$. Then $P$ is the only Sylow $p$-subgroup of $G$.

(2) In any finite group $G$, $P$ is the only Sylow $p$-subgroup of $N_G(P)$.

**Theorem 7.5. (Sylow’s Third Theorem)** Let $P$ be a Sylow $p$-subgroup of $G$. Then the number of Sylow $p$-subgroups of $G$ is $(G : N_G(P))$. Further, $(G : N_G(P)) \equiv 1$ (mod $p$).

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### Week 8 — Lecture 19 — Friday 8th May.

**Theorem 7.6. (Cauchy’s Theorem)** Let $p$ divide $|G|$. Then $G$ contains an element of order $p$.

**Corollary 7.7.** If $p$ divides $|G|$ then $G$ has a subgroup of order $p$.

#### 7.4. Examples. We consider the structure of groups of order $pq$, where $p$ and $q$ are distinct odd primes, and groups of order $2p$. 