School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 8 Summary

The Lemma below was actually proved in the Friday lecture of Week 7.

Lemma 7.2. *Let G be a finite p-group acting on the finite set X. Let*

$$F = \{x \in X \mid g * x = x \text{ for all } g \in G\}.$$

Then $|F| \equiv |X| \pmod{p}$.

Week 8 — Lecture 18 — Tuesday 5th May.

7.3. **Sylow's second and third theorems.**

Theorem 7.3. (Sylow's Second Theorem) *Let P be a Sylow p-subgroup of the finite group G of order $p^m r$, where r is coprime to p. If Q is any p-subgroup of G (that is, $|Q|$ is a power of p) then $Q < gPg^{-1}$ for some $g \in G$.*

In particular, all Sylow p-subgroups are conjugate.

Lemma 7.4.

(1) *Let P be a Sylow p-subgroup of G and suppose $P \lhd G$. Then P is the* only *Sylow p-subgroup of G.*
(2) *In any finite group G, P is the only Sylow p-subgroup of $N_G(P)$.*

Theorem 7.5. (Sylow's Third Theorem) *Let P be a Sylow p-subgroup of G. Then the number of Sylow p-subgroups of G is $(G : N_G(P))$. Further, $(G : N_G(P)) \equiv 1 \pmod{p}$.*

Week 8 — Lecture 19 — Friday 8th May.

Theorem 7.6. (Cauchy's Theorem) *Let p divide $|G|$. Then G contains an element of order p.*

Corollary 7.7. *If p divides $|G|$ then G has a subgroup of order p.*

7.4. **Examples.** We consider the structure of groups of order pq, where p and q are distinct odd primes, and groups of order $2p$.