School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 7 Summary

This material was covered in Lecture 14

---

6. Groups Acting on Sets

6.1. Introduction.

Definition 6.1. Let G be a group and X a set. An *action of G on X* is a map $G \times X \to X$, $(g, x) \mapsto g * x$ such that

(i) for each $g_1, g_2 \in G$ and $x \in X$,

$$(g_1 g_2) * x = g_1 * (g_2 * x)$$

(ii) for each $x \in X$, $e * x = x$.

If there is no confusion, we may write gx for $g * x$.

Note:
(1) S_n acts on $X = \{1, 2, ..., n\}$.
(2) G acts on $X = G$ by
  (a) conjugation: $g * x = gxg^{-1}$
  (b) left multiplication: $g * x = gx$.
(3) If $H < G$, G acts on the left cosets of H by left multiplication: $g * xH = gxH$.
(4) If $G = GL(n, F)$ and V is a vector space of dimension n over F, then G acts on V by matrix multiplication.

Definition 6.2. If G acts on X then for any $x \in X$, $[x] = \{g * x \mid g \in G\}$ is called an *orbit* in X of the action.

If there is only one orbit then we say G is *transitive* on X.

Week 6 — Lecture 15 — Tuesday 28th April.

Proposition 6.3. *The orbits of a group G acting on a set X are the equivalence classes under the equivalence relation on X:*

$$x \sim y \text{ if and only if } y = g * x \text{ for some } g \in G.$$

Hence X is the disjoint union of the distinct orbits.

Definition 6.4. If G acts on X then for any $x \in X$, the *stabilizer* of $x \in X$ is

$$S_G(x) = \{g \in G \mid g * x = x\}.$$

The stabilizer of x is a subgroup of G. It is sometimes called the *isotropy subgroup* of x, and sometimes denoted G_x.

Week 6 — Lecture 16 — Wednesday 29th April.

6.2. The Orbit-Stabilizer Theorem.

Theorem 6.5. (Orbit-Stabilizer Theorem) *Let G act on X. Then for any $x \in X$,*

$$|[x]| = (G : S_G(x)).$$

6.3. Burnside's Theorem.

Theorem 6.6. (Burnside's Theorem) *Let G be a finite group and X a finite set such that G acts on X. Let r be the number of distinct orbits of G on X and for each $g \in G$ let*

$$X_g = \{x \in X \mid g * x = x\},$$

the set of all elements in X fixed by g. Then

$$r|G| = \sum_{g \in G} |X_g|.$$

6.4. Cayley's Theorem.

Theorem 6.7. (Cayley's Theorem) *Every group is isomorphic to a group of permutations.*

6.4.1. *Application of Burnside's theorem to chemistry.*

Week 6 — Lecture 17 — Friday 1st May.

7. The Sylow Theorems

7.1. Sylow's first theorem. The results of this chapter are due to the Norwegian mathematician Ludvig Sylow (1832 – 1918), though the proofs have been modernized. Along with Lagrange's theorem, they are the most important results of finite group theory – Lagrange's theorem gives a necessary condition for subgroups, and Sylow's theorems give sufficient conditions.

Theorem 7.1. Sylow's First Theorem *Let G be a finite group of order $p^m r$, where p is a prime and r is coprime to p. Then G has a subgroup P of order p^m.*

Such a subgroup P, the existence of which is guaranteed by this theorem, is called a *Sylow p-subgroup* of G.