To determine the rank and torsion invariants of $G$ we use the following procedure.

Write the exponents $n_{ij}$ in a matrix $N$, with the $j$th relation corresponding to the $j$th column. There must be at least as many columns as rows, so we have an $m \times n$ matrix with $n \geq m$. (If not, add columns of zeros to make $n \geq m$).

We then use certain row and column operations to reduce $N$ to a diagonal matrix in which the diagonal entries are $d_1, \ldots, d_t, 0, \ldots, 0$ and the successive non-zero entries divide one another: $d_1 \mid d_2 \mid \ldots \mid d_t$. Then the entries $d_1, \ldots, d_t$ are the torsion invariants of $G$ and the number of zeros is the rank of $G$.

5.2.1. **Permissible row and column operations.**

(i) Interchange any two rows: $R_i \sim R_j$, $R_j \sim R_i$.
(ii) Multiply any row by $-1$: $R_i \sim -R_i$.
(iii) Add to any row an integer multiple of another row: $R_i \sim R_i + cR_j$, $c \in \mathbb{Z}$.

The corresponding column operations are also permitted.

It is not permissible to:

(i) Multiply a row by $c$, if $c \neq \pm 1$.
(ii) Replace $R_i$ by $cR_i + dR_j$, if $c \neq \pm 1$.

5.2.2. **Why does it work?** Row operations correspond to changing the generators, column operations to manipulating the relations. Specifically, the row operation $R_i \sim R_i + cR_j$ corresponds to replacing generator $x_j$ by $y_j = x_j x_i^{-c}$.

5.2.3. **Procedure.** The initial aim is to get the g.c.d. of all entries in the matrix to the $(1, 1)$ position, and then use this entry as a pivot to eliminate all other entries in the first row and column. Then repeat this procedure on the $(m-1) \times (n-1)$ submatrix obtained by removing the first row and column. Continue.

To get the g.c.d. to the $(1, 1)$ position, it will in general be necessary to use the Division Algorithm several times on the rows and/or columns, as in the following examples:

\[
\begin{bmatrix}
7 & 30 \\
0 & 20
\end{bmatrix}
\sim
\begin{bmatrix}
7 & \vdots \\
2 & \vdots
\end{bmatrix}
(R_2 \sim R_2 - 4R_1)
\sim
\begin{bmatrix}
1 & 2 \\
5 & 20
\end{bmatrix}
(R_1 \sim R_1 - 3R_2).
\]

Example 1: Find the torsion invariants of the abelian group $C_8 \times C_{20}$.

Solution:

$C_8 \times C_{20} \cong C_8 \times C_4 \times C_5 \cong C_4 \times C_{40}$. The torsion invariants are 4 and 40.

Example 2: Find the torsion invariants and rank of the abelian group

\[G = \langle a, b, c, d \mid a^{-3} b c^6 d^4 = a^{-1} b^3 c^2 d^4 = a^{14} b^{-2} c^{-8} d^4 = 1 \rangle.\]

Solution:
that is, $C_1 \times C_8 \times C_{20} \times C_\infty$, but this is not yet in the required form. Continuing, the matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 \\
0 & 20 & 20 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 \\
0 & 4 & 20 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 \\
0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 4 & 20 & 0 \\
0 & 0 & -40 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 40 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

So the torsion invariants are 4 and 40, and the rank is 1.