School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 6 Summary

Week 6 — Lecture 14 — Tuesday 7th April.

To determine the rank and torsion invariants of G we use the following procedure.

Write the exponents n_{ij} in a matrix N, with the jth relation corresponding to the jth *column*. There must be at least as many columns as rows, so we have an $m \times n$ matrix with $n \geq m$. (If not, add columns of zeros to make $n \geq m$).

We then use certain row and column operations to reduce N to a diagonal matrix in which the diagonal entries are $d_1, ..., d_t, 0, ..., 0$ and the successive non-zero entries divide one another: $d_1 \mid d_2 \mid ... \mid d_t$. Then the entries $d_1, ...d_t$ are the torsion invariants of G and the number of zeros is the rank of G.

5.2.1. *Permissible row and column operations.*

(i) Interchange any two rows: $R_i,\ R_j \rightsquigarrow R_j,\ R_i$.
(ii) Multiply any row by -1: $R_i \rightsquigarrow -R_i$.
(iii) Add to any row an integer multiple of another row: $R_i \rightsquigarrow R_i + cR_j,\ c \in \mathbb{Z}$.

The corresponding column operations are also permitted.

It is not permissible to:
(i) Multiply a row by c, if $c \neq \pm 1$.
(ii) Replace R_i by $cR_i + dR_j$, if $c \neq \pm 1$.

5.2.2. *Why does it work?* Row operations correspond to changing the generators, column operations to manipulating the relations. Specifically, the row operation $R_i \rightsquigarrow R_i + cR_j$ corresponds to replacing generator x_j by $y_j = x_j x_i^{-c}$.

5.2.3. *Procedure.* The initial aim is to get the g.c.d. of all entries in the matrix to the $(1,1)$ position, and then use this entry as a pivot to eliminate all other entries in the first row and column. Then repeat this procedure on the $(m-1) \times (n-1)$ submatrix obtained by removing the first row and column. Continue.

To get the g.c.d. to the $(1,1)$ position, it will in general be necessary to use the Division Algorithm several times on the rows and/or columns, as in the following examples:

$$\begin{bmatrix} 7 & \cdots \\ 30 & \cdots \end{bmatrix} \sim \begin{bmatrix} 7 & \cdots \\ 2 & \cdots \end{bmatrix} (R_2 \rightsquigarrow R_2 - 4R_1) \sim \begin{bmatrix} 1 & \cdots \\ 2 & \cdots \end{bmatrix} (R_1 \rightsquigarrow R_1 - 3R_2).$$

$$\begin{bmatrix} 15 & 0 \\ 0 & 20 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 \\ 20 & 20 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 \\ 5 & 20 \end{bmatrix} \sim \begin{bmatrix} 5 & 20 \\ 15 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 \\ 0 & -60 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 \\ 0 & 60 \end{bmatrix}.$$

Example 1: Find the torsion invariants of the abelian group $C_8 \times C_{20}$.

Solution:

$C_8 \times C_{20} \simeq C_8 \times C_4 \times C_5 \simeq C_4 \times C_{40}$. The torsion invariants are 4 and 40.

Example 2: Find the torsion invariants and rank of the abelian group

$$G = \langle a, b, c, d \mid a^{-3}bc^6d^4 = a^{-1}b^3c^2d^4 = a^{14}b^{-2}c^{-8}d^4 = 1 \rangle.$$

Solution:

$$\begin{bmatrix} -3 & -1 & 14 & 0 \\ 1 & 3 & -2 & 0 \\ 6 & 2 & -8 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & -16 & 4 & 0 \\ 0 & -8 & 12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

that is, $C_1 \times C_8 \times C_{20} \times C_\infty$, but this is not yet in the required form. Continuing, the matrix is

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 20 & 20 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 4 & 20 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 20 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 20 & 0 \\ 0 & 0 & -40 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 40 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the torsion invariants are 4 and 40, and the rank is 1.