School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 5 Summary

Week 5 — Lecture 11 — Tuesday 31 March.

4.2. **Direct products of groups.** Let H and K be groups. Then we can make the cartesian product

$$H \times K = \{(h, k) \mid h \in H, k \in K\}$$

into a group, called the *(external) direct product* of H and K, by defining

$$(h, k) \cdot (h', k') = (hh', kk')$$

for all $h, h' \in H, k, k' \in K$. Then $H \times K$ has subgroups

$$\begin{aligned} H_0 &= \{(h, e) \mid h \in H\} \simeq H, \\ K_0 &= \{(e, k) \mid k \in K\} \simeq K. \end{aligned}$$

Proposition 4.2. *Let H and K be groups as above. Then:*

(1) $H_0 \cap K_0 = \{(e, e)\} = \{e\}$.
(2) For all $h \in H, k \in K$ we have $(h, e) \cdot (e, k) = (h, k) = (e, k) \cdot (h, e)$. Hence $G = H_0K_0$.
(3) We write (h, e) as h and (e, k) as k, and identify H_0 and K_0 with H and K. Then every $g \in G$ can be written uniquely as $g = hk$ for $h \in H, k \in K$.
(4) $H \lhd G$ *and* $K \lhd G$.
(5) $|G| = |H \times K| = |H|.|K|$.
(6) $G/H \simeq K$ *and* $G/K \simeq H$.

Week 5 — Lecture 12 — Wednesday 1 April.

4.3. **The internal direct product.**

Definition 4.3. A group G is *decomposable* if it is isomorphic to a direct product of two proper non-trivial subgroups. Otherwise G is indecomposable.

If G is decomposable then G has subgroups H and K such that

(i) $H \cap K = \{e\}$
(ii) $G = HK$
(iii) $hk = kh$ for all $h \in H, k \in K$.

Then we write $G = H \times K$ and say that G is the *(internal) direct product* of H and K.

Equivalently, if (iii)$'$ is the statement:

(iii)$'$ $H \lhd G$ and $K \lhd G$

then (i), (ii) and (iii)$'$ imply that $G = H \times K$.

5. Finitely generated abelian groups

5.1. The fundamental theorem.

Definition 5.1. A group G is *finitely generated* if there is some finite subset X of G such that $G = \langle X \rangle$.

Thus $G = \langle x_1, ..., x_n \rangle$, the set of all finite products of the x_is and their inverses.

Definition 5.2. If every element of a group G has finite order then G is called a *torsion group*. If only the identity e has finite order then G is called a *torsion-free group*. If G is an abelian group, then the subgroup of G consisting of all elements of finite order is called the *torsion subgroup* of G and denoted $\mathrm{Tor}(G)$.

Theorem 5.3. (Fundamental Theorem of Finitely Generated Abelian Groups) *Every finitely generated abelian group is isomorphic to a direct product of cyclic groups of the form*

$$C_{n_1} \times C_{n_2} \times ... \times C_{n_s} \times C_\infty \times ... \times C_\infty,$$

where each $n_i = p_i^{a_i}$ for some prime p_i and $a_i \in \mathbb{N}$. (The p_i need not be distinct.)

Week 5 — Lecture 13 — Friday 3 April.

Note:

(1) The torsion subgroup of G is $\mathrm{Tor}(G) = C_{n_1} \times C_{n_2} \times ... \times C_{n_s}$. Thus $|T| = n_1 n_2 ... n_s$.

(2) The group $F = \underbrace{C_\infty \times ... \times C_\infty}_{f \text{ factors}}$ is torsion free. (It is called a free abelian group of rank f.) The number of factors f is the *(free) rank* or *Betti number* of G. G is finite if and only if $f = 0$.

(3) Since $C_n \times C_m \simeq C_{nm}$ if m and n are coprime, we can also write

$$T \simeq C_{d_1} \times ... \times C_{d_t}$$

where $d_1 \mid d_2 \mid ... \mid d_t$ and $|T| = d_1 d_2 ... d_t$. The d_i, known as the *torsion invariants* of G, are unique.

(4) Two finitely generated abelian groups are isomorphic if and only if they have the same free rank and the same torsion invariants.

Corollary 5.4. *The indecomposable finite abelian groups are precisely the cyclic groups of order p^a, where p is prime, $a \in \mathbb{N}$.*

Corollary 5.5. *If G is a finite abelian group and m divides $|G|$ then G has a subgroup of order m.*

5.2. Generators and relations for abelian groups.

Suppose that an abelian group is defined by generators $x_1, x_2, ..., x_m$ and a number of relations of the form

$$\begin{array}{rcl} x_1^{n_{11}} x_2^{n_{21}} \dots x_m^{n_{m1}} & = & e \\ x_1^{n_{12}} x_2^{n_{22}} \dots x_m^{n_{m2}} & = & e \\ \vdots & & \vdots \\ x_1^{n_{1n}} x_2^{n_{2n}} \dots x_m^{n_{mn}} & = & e. \end{array}$$

We also know that $[x_i, x_j] = e$ for all i, j as G is abelian.