

School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 4 Summary

Week 4 — Lecture 9 — Tuesday 24th March.

3.3. Related results.

Lemma 3.6. Let G be a group such that $G/Z(G)$ is cyclic. Then G is abelian.

Corollary 3.7. $G/Z(G)$ cannot be cyclic of order greater than one.

Lemma 3.8. Every group of order p^2 is abelian.

Theorem 3.9. Let $N \triangleleft G$. Then there is a 1-1 correspondence between subgroups of G containing N and subgroups of G/N , namely

$$\text{if } N < H < G \text{ then } H \leftrightarrow H/N.$$

Every subgroup of G/N is of form H/N for some subgroup H of G containing N .

Also, $H \triangleleft G$ if and only if $H/N \triangleleft G/N$.

3.4. Composition series.

Definition 3.10. Let $N \triangleleft G$. Then N is called a *maximal normal subgroup* of G if the only normal subgroup of G that properly contains N is G itself.

Then N is a maximal normal subgroup of G if and only if G/N is simple.

Definition 3.11. A *composition series* of a group G is a sequence of subgroups

$$\{e\} = N_{k+1} \triangleleft N_k \triangleleft \dots \triangleleft N_2 \triangleleft N_1 \triangleleft N_0 = G,$$

such that each N_{i+1} is a maximal normal subgroup of N_i . That is, each factor group N_i/N_{i+1} is simple.

Theorem 3.12. The Jordan-Hölder Theorem states that for any composition series, the number of factors k and the set of factor groups $\{N_i/N_{i+1} \mid i = 0, 1, \dots, k\}$ is unique.

Week 4 — Lecture 10 — Friday 27th March.

3.5. **The derived group.** Let X be a subset of G . Then $H = \langle X \rangle$ denotes the smallest subgroup of G containing X . We say that H is *generated* by X . Then H is the set of all products of the form $x_i^{n_i} \dots x_j^{n_j}$, where $x_i, \dots, x_j \in X$ and $n_i, \dots, n_j \in \mathbb{Z}$.

Definition 3.13. The commutator of the elements $g, h \in G$ is $[g, h] = ghg^{-1}h^{-1}$. The *derived group* or *commutator subgroup* of G is the group

$$G' = [G, G] = \langle [g, h] \mid g, h \in G \rangle.$$

Note 3.1.

- (1) Elements g and h commute if and only if $[g, h] = e$.
- (2) $[g, h]^{-1} = [h, g]$.
- (3) $G' = \{e\}$ if and only if G is abelian.

Proposition 3.14. Let G be a group and G' its commutator subgroup. Then:

- (a) $G' \triangleleft G$.
- (b) G/G' is abelian.
- (c) If $N \triangleleft G$ and G/N is abelian, then $G' \triangleleft N$. Thus G' is the smallest normal subgroup of G with abelian factor group.

4. PRODUCTS OF GROUPS

4.1. **The isomorphism theorem.** Let H and K be subgroups of the group G . We define

$$HK = \{hk \mid h \in H, k \in K\}.$$

Then $HK < G$ if and only if $HK = KH$.

In particular, if $H \triangleleft G$ or $K \triangleleft G$ then $HK < G$.

If $HK < G$, then

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Theorem 4.1. (The Isomorphism Theorem) Let H and K be subgroups of G with $H \triangleleft G$. Then $HK/H \simeq K/H \cap K$.