School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 4 Summary

Week 4 — Lecture 9 — Tuesday 24th March.

3.3. **Related results.**

Lemma 3.6. *Let G be a group such that $G/Z(G)$ is cyclic. Then G is abelian.*

Corollary 3.7. *$G/Z(G)$ cannot be cyclic of order greater than one.*

Lemma 3.8. *Every group of order p^2 is abelian.*

Theorem 3.9. *Let $N \lhd G$. Then there is a 1-1 correspondence between subgroups of G containing N and subgroups of G/N, namely*

$$\text{if } N < H < G \text{ then } H \leftrightarrow H/N.$$

Every subgroup of G/N is of form H/N for some subgroup H of G containing N.

Also, $H \lhd G$ if and only if $H/N \lhd G/N$.

3.4. **Composition series.**

Definition 3.10. Let $N \lhd G$. Then N is called a *maximal normal subgroup* of G if the only normal subgroup of G that properly contains N is G itself.

Then N is a maximal normal subgroup of G if and only if G/N is simple.

Definition 3.11. A *composition series* of a group G is a sequence of subgroups

$$\{e\} = N_{k+1} \lhd N_k \lhd \ldots \lhd N_2 \lhd N_1 \lhd N_0 = G,$$

such that each N_{i+1} is a maximal normal subgroup of N_i. That is, each factor group N_i/N_{i+1} is simple.

Theorem 3.12. *The* Jordan-Hölder Theorem *states that for any composition series, the number of factors k and the set of factor groups $\{N_i/N_{i+1} \mid i = 0, 1, \ldots, k\}$ is unique.*

Week 4 — Lecture 10 — Friday 27th March.

3.5. **The derived group.** Let X be a subset of G. Then $H = \langle X \rangle$ denotes the smallest subgroup of G containing X. We say that H is *generated* by X. Then H is the set of all products of the form $x_i^{n_i} \ldots x_j^{n_j}$, where $x_i, \ldots, x_j \in X$ and $n_i, \ldots, n_j \in \mathbb{Z}$.

Definition 3.13. The commutator of the elements $g, h \in G$ is $[g, h] = ghg^{-1}h^{-1}$. The *derived group* or *commutator subgroup* of G is the group

$$G' = [G, G] = \langle [g, h] \mid g, h \in G \rangle.$$

Note 3.1.

(1) Elements g and h commute if and only if $[g, h] = e$.
(2) $[g, h]^{-1} = [h, g]$.
(3) $G' = \{e\}$ if and only if G is abelian.

Proposition 3.14. *Let G be a group and G' its commutator subgroup. Then:*

(a) $G' \lhd G$.
(b) G/G' is abelian.
(c) If $N \lhd G$ and G/N is abelian, then $G' < N$. Thus G' is the smallest normal subgroup of G with abelian factor group.

4. Products of Groups

4.1. The isomorphism theorem. Let H and K be subgroups of the group G. We define

$$HK = \{hk \mid h \in H, k \in K\}.$$

Then $HK < G$ if and only if $HK = KH$.

In particular, if $H \lhd G$ *or* $K \lhd G$ then $HK < G$.

If $HK < G$, then

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Theorem 4.1. (The Isomorphism Theorem) *Let H and K be subgroups of G with $H \lhd G$. Then $HK/H \simeq K/H \cap K$.*