School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 3 Summary

Week 3 — Lecture 6 — Tuesday 17th March.

Note 2.2.

(1) A conjugate of x has the same order as x. (Assignment 1)
(2) We say that x is *conjugate to* y if y is a conjugate of x, ie if there is some $g \in G$ with $y = gxg^{-1}$.

Proposition 2.10. *Conjugacy is an equivalence relation on G.*

Note 2.3. The equivalence class of x is called the *conjugacy class* of x and denoted $[x]$. The conjugacy classes partition G:

$$G = [1] \cup [x] \cup \ldots \cup [z].$$

2.2.1. *Centralizer.*

Definition 2.11. The *centralizer* $C_G(x)$ of x in G is the subgroup consisting of all elements of G that commute with x.

Thus, $C_G(x) = \{g \in G \mid gx = xg\} = \{g \in G \mid gxg^{-1} = x\}$.

Note 2.4.

(1) $\langle x \rangle < C_G(x)$.
(2) If G is abelian, then $C_G(x) = G$.

Proposition 2.12. *If $x \in G$ a finite group then $|[x]| = (G : C_G(x))$.*

2.2.2. *Centre.*

Definition 2.13. The *centre* $Z(G)$ of a group G is the subgroup of G consisting of all elements $x \in G$ that commute with *every* elements of G.

Thus, $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$.

Note:

(1) $Z(G) \lhd G$.
(2) $Z(G) = G$ if and only if G is abelian.
(3) $x \in Z(G)$ if and only if $[x] = \{x\}$, or equivalently $|[x]| = 1$.

2.2.3. *Simple groups.*

Definition 2.14. A group G is called *simple* if G has no proper non-trivial normal subgroups.

Week 3 — Lecture 7 — Wednesday 18th March.

Theorem 2.15. *An abelian simple group G with $|G| > 1$ must be isomorphic to C_p for some prime p.*

Definition 2.16. A group of order p^n, where p is prime, is called a *p-group*.

Lemma 2.17. *Let P be a p-group of order p^n, $n \geq 1$. Then $Z(P) \neq \langle e \rangle$. Thus P is not simple unless $n = 1$, that is $P \simeq C_p$.*

2.2.4. *Conjugates of a subgroup, and the normalizer.* If $H < G$, the conjugates of H are the subgroups gHg^{-1}, for $g \in G$.

Definition 2.18. The *normalizer* of a subgroup H of G is the subgroup

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Note 2.5. $N_G(H)$ is the largest subgroup of G in which H is normal. That is if $H \triangleleft N_G(H)$, and if $H \triangleleft K < G$ then $K < N_G(H)$.

Proposition 2.19. *If H is a subgroup of a finite group G then the number of distinct conjugates of H in G equals $(G : N_G(H))$.*

Week 3 — Lecture 8 — Friday 20th March.

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3. Homomorphisms and Factor Groups

3.1. Homomorphisms.

Definition 3.1. If G and H are groups, a *homomorphism* from G to H is a function $f : G \to H$ such that

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$.

Proposition 3.2. *If $f : G \to H$ is a homomorphism, then*

(1) $f(e) = e$.
(2) $f(g^{-1}) = (f(g))^{-1}$.
(3) The image *of f,*

$$\mathrm{im}(f) = f(G) = \{f(g) \mid g \in G\},$$

is a subgroup of H.
(4) The kernel *of f,*

$$\ker f = \{g \in G \mid f(g) = e\},$$

is a normal subgroup of G.
(5) A homomorphism f is one to one if and only if $\ker f = \langle e \rangle$. *So f is an isomorphism if and only if* $\ker f = \{e\}$ *and* $\mathrm{im}(f) = H$.

3.2. The factor group.

Let $N \triangleleft G$. Consider the set

$$G/N = \{gN \mid g \in G\}$$

of left cosets of N in G. This set is a group under the operation

$$gNhN = (gh)N.$$

This group is called the *factor or quotient group* of G by N. Its order is $|G|/|N| = (G : N)$.

Theorem 3.3. (Homomorphism Theorem) *Let $f : G \to H$ be a homomorphism. Then the groups $G/\ker f$ and $f(G)$ are isomorphic.*

Theorem 3.4. *Let $N \triangleleft G$. Then the function $f : G \to G/N$ given by $f(g) = gN$ is a homomorphism with kernel N.*