School of Mathematical Sciences
PURE MTH 3007
Groups and Rings III, Semester 1, 2009

Week 2 Summary

Definition 1.16. A *permutation group of degree* n is a subgroup of S_n.

Week 2 — Lecture 4 — Tuesday 10th March.

1.2.3. *Symmetry groups.* The symmetries of the square form a group of order 8, the *dihedral* group D_4. Similarly, the symmetries of the regular n-gon form a group of order $2n$, the nth dihedral group D_n. Clearly $D_n < S_n$, so D_4 is another example of a permutation group of degree 4.

1.3. Isomorphism.

Definition 1.17. Two groups G and H are called *isomorphic* if there is a $1-1$, onto function $\phi\colon G \to H$ such that for all $x, y \in G$ we have $\phi(xy) = \phi(x)\phi(y)$.

Note 1.1. We call such a ϕ an isomorphism. If G and H are isomorphic, we write $G \simeq H$.

Proposition 1.18. *Assume that $\phi\colon G \to H$ is an isomorphism and that $x \in G$. Denote the identities of G and H by e_G and e_H. Then*
(a) $\phi(e_G) = e_H$.
(b) $\phi(x^{-1}) = (\phi(x))^{-1}$
(c) $|G| = |H|$
(d) Either x and $\phi(x)$ are both of infinite order or they have equal finite order.
(e) If G is abelian so is H.

2. Cosets and Normal Subgroups

2.1. Cosets.

Definition 2.1. Let $H < G$. A *left coset* of H in G is a set of the form
$$xH = \{xh \mid h \in H\},$$
where x is an element of G. Similarly, a *right coset* is a set of the form
$$Hx = \{hx \mid h \in H\}.$$

Proposition 2.2. *Let $H < G$. Then*
(a) $|gH| = |H| = |Hg|$.
(b) If $x, y \in G$ then either $x^{-1}y \in H$ and $xH = yH$ or $x^{-1}y \notin H$ and $xH \cap yH = \emptyset$.
(c) If x, y inG then either $yx^{-1} \in H$ and $Hx = Hy$ or $yx^{-1} \notin H$ and $xHx \cap Hy = \emptyset$.
(d) Every element of G is in exactly one left coset of H and exactly one right coset of H.
(e) G is the disjoint union of the left (or right) cosets of H.

Definition 2.3. If $H < G$, the *index* of H in G is the number of distinct left cosets of H in G. It is denoted $(G : H)$.

Theorem 2.4. *(Lagrange's Theorem) If H is a subgroup of a finite group G then*
$$(G : H) = \frac{|G|}{|H|}$$
and thus $|H|$ divides $|G|$.

Corollary 2.5. *If x is an element of the finite group G, then $|x|$ divides $|G|$.*

Corollary 2.6. *Every group of prime order is cyclic.*

Week 2 — Lecture 5 — Friday 13th March.

2.2. **Normal subgroups.** If $H < G$ and $g \in G$, the left coset gH and the right coset Hg are in general not the same set. For example, consider $G = S_3 = \{1, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$ and the subgroup $H = \{1, (1\,2)\}$.

Left cosets of H	Right cosets of H
$1H = \{1, (1\ 2)\}$	$H1 = \{1, (1\ 2)\}$
$(1\ 2)H = \{(1\ 2), 1\}$	$H(1\ 2) = \{(1\ 2), 1\}$
$(1\ 3)H = \{(1\ 3), (1\ 2\ 3)\}$	$H(1\ 3) = \{(1\ 3), (1\ 3\ 2)\}$
$(2\ 3)H = \{(2\ 3), (1\ 3\ 2)\}$	$H(2\ 3) = \{(2\ 3), (1\ 2\ 3)\}$
$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$	$H(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3)\}$
$(1\ 3\ 2)H = \{(1\ 3\ 2), (2\ 3)\}$	$H(1\ 3\ 2) = \{(1\ 3\ 2), (1\ 3)\}$

Compare this example with what we get when we consider the subgroup $A_3 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$:

Left cosets of A_3	Right cosets of A_3
$1A_3 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$	$A_31 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$
$(1\ 2)A_3 = \{(1\ 2), (2\ 3), (1\ 3)\}$	$A_3(1\ 2) = \{(1\ 2), (1\ 3), (2\ 3)\}$
$(1\ 3)A_3 = \{(1\ 3), (1\ 2), (2\ 3)\}$	$A_3(1\ 3) = \{(1\ 3), (2\ 3), (1\ 2)\}$
$(2\ 3)A_3 = \{(2\ 3), (1\ 3), (1\ 2)\}$	$A_3(2\ 3) = \{(2\ 3), (1\ 2), (1\ 3)\}$
$(1\ 2\ 3)A_3 = \{(1\ 2\ 3), (1\ 3\ 2), 1\}$	$A_3(1\ 2\ 3) = \{(1\ 2\ 3), (1\ 3\ 2), 1\}$
$(1\ 3\ 2)A_3 = \{(1\ 3\ 2), 1, (1\ 2\ 3)\}$	$A_3(1\ 3\ 2) = \{(1\ 3\ 2), 1, (1\ 2\ 3)\}$

We see that $gA_3 = A_3\, g$ for every $g \in A_3$.

Definition 2.7. A subgroup H of a group G is *normal* if for all $g \in G$, $gHg^{-1} = H$.

We write $H \lhd G$. Equivalently, $H \lhd G$ if $gH = Hg$ for all $g \in G$.

Notes:

(1) We saw in the above examples that $\{1, (1\ 2)\} \not\lhd S_3$ and $A_3 \lhd S_3$.
(2) Whenever $(G : H) = 2$, $H \lhd G$. In particular, $A_n \lhd S_n$ for $n = 3, 4, 5, \ldots$.
(3) Every subgroup of an abelian group is normal.
(4) $\{1\} \lhd G$ and $G \lhd G$.
(5) If $H \lhd G$ and $K \lhd G$ then $H \cap K \lhd G$.
(6) If $N \lhd G$ and $N < H < G$ then $N \lhd H$.

2.3. **Conjugation.**

Definition 2.8. Let $g \in G$ and let $X \subset G$. Then the subset gXg^{-1} is called a *conjugate* of X in G. In particular, if $x \in G$, then the element gxg^{-1} is called a *conjugate* of x (in G).