We are about a lecture ahead of where I was planning to be so I will start recalibrating the lecture plan.

On Friday of Week 9 we covered:

10.3. **Polynomial functions.** Let \( R \) be a ring and \( f(x) = a_0 + a_1x + a_2x^2 + \cdots \) a polynomial over \( R \). Then the function \( \overline{f} : R \to R \) given by \( \overline{f}(r) = a_0 + a_1r + a_2r^2 + \cdots \) is called the **polynomial function** associated to \( f \).

The set \( \mathcal{P}(R) \) of all polynomial functions over \( R \) is a ring under the operations \((\overline{f} + \overline{g})(r) = \overline{f}(r) + \overline{g}(r)\) and \((\overline{f\cdot g})(r) = \overline{f}(r)\cdot \overline{g}(r)\). It is then easy to show that

\[
\overline{f + g} = \overline{f} + \overline{g}, \quad \overline{f \cdot g} = \overline{f} \cdot \overline{g}.
\]

If \( R \) is a commutative ring with identity then so is \( \mathcal{P}(R) \), but note that \( \mathcal{P}(R) \) is not necessarily isomorphic to \( R[x] \).

10.3.1. **Zeros of polynomials.** Let \( F \) be a field.

**Definition 10.3.** An element \( a \in F \) is a **zero** of \( f(x) \in F[x] \) if \( \overline{f}(a) = 0 \).

**Theorem 10.4** (Factor Theorem). The element \( a \in F \) is a zero of \( f(x) \in F[x] \) if and only if \( x - a \mid f(x) \).

**Corollary 10.5.** A polynomial of degree \( n \) over a field \( F \) has at most \( n \) zeros in \( F \).

**Definition 10.6.** A non-constant polynomial \( f(x) \in F[x] \) is **irreducible** over \( F \) if \( f(x) \neq g(x)h(x) \) for any polynomials \( g(x), h(x) \) of degree less than \( f(x) \).

11. **IDEALS**

11.1. **Introduction.**

**Definition 11.1.** A subring \( I \) of a ring \( R \) is called an **ideal** of \( R \) if for all \( r \in R \) and \( i \in I \) we have \( ir \in I \) and \( ri \in I \).

11.2. **The Factor Ring.**

**Theorem 11.2** (The Factor Ring). Let \( I \) be an ideal of the ring \( R \). Then the set \( R/I \) of all cosets of \( I \) in \( R \) is a ring under the operations

\[
(r + I) + (s + I) = (r + s) + I \quad \text{and} \quad (r + I) \cdot (s + I) = rs + I.
\]

If \( R \) is a commutative ring, or a ring with identity, then so is \( R/I \).

**Lemma 11.3.** Let \( \phi : R \to S \) be a ring homomorphism. Then \( \ker \phi \) is an ideal of \( R \).

**Theorem 11.4** (Homomorphism Theorem). If \( \phi : R \to S \) is a ring homomorphism then \( R/\ker \phi \cong \phi(R) \).

**Lemma 11.5.** If \( I \) and \( J \) are ideals of \( R \) then so are \( I + J \) and \( I \cap J \).
Theorem 11.6 (Isomorphism Theorem).

(i) Let $I$ be an ideal of $R$. Then there is a $1-1$ correspondence between subrings $S$ of $R$ containing $I$ and subrings $S/I$ of $R/I$. Here $S$ is an ideal of $R$ if and only if $S/I$ is an ideal of $R/I$.

(ii) Let $I \subset J \subset R$ with $I$ and $J$ ideals of $R$. Then

$$R/J \simeq (R/I)/(J/I).$$

(iii) Let $I$ and $J$ be ideals of $R$. Then

$$(I + J)/J \simeq I/(I \cap J).$$

Week 10 — Lecture 26 — Friday 22nd May.

11.3. Ideals in commutative rings with identity. Let $R$ be a commutative ring with identity.

Definition 11.7. An ideal of the form $\langle a \rangle = \{ar \mid r \in R\}$ is called a principal ideal of $R$.

An ideal $M$ of $R$ is called a maximal ideal if there is no ideal $I$ of $R$ such that $M \subset I \subset R$.

Theorem 11.8. Let $R$ be a commutative ring with identity. Then $M$ is a maximal ideal of $R$ if and only if $R/M$ is a field.

12. Factorization in Integral Domains

12.1. Irreducibles and associates.

Definition 12.1. An element $c$ of an integral domain, not zero or a unit, is called irreducible if, whenever $c = df$, one of $d$ or $f$ is a unit.

Elements $c$ and $d$ are called associates if $c = du$ for a unit $u$.

12.2. Euclidean domains.

Definition 12.2. A Euclidean domain is an integral domain $D$ together with a function $\delta : D^* \rightarrow \mathbb{N}$ satisfying

(i) $\delta(a) \leq \delta(ab)$ for all non-zero $a, b \in D$

(ii) for all $a, b \in D, b \neq 0$ there exist $q, r \in D$ such that

$$a = bq + r$$

with either $r = 0$ or $\delta(r) < \delta(b)$.

The function $\delta$ is called a Euclidean valuation.

Examples:

1. $\mathbb{Z}$ with $\delta(n) = |n|$.
2. $F[x]$ with $\delta(f(x)) = \deg f(x)$, where $F$ is a field.