Examination in School of Mathematical Sciences

Semester 1, 2008

004094	Groups and Rings III
	PURE MTH 3007

Official Reading Time: 10 mins
Writing Time: 180 mins
Total Duration: 190 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 100

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Give a True (T) or False (F) answer to each of the following statements. In each case give a *short* reason for your answer.
   (a) Every abelian group of order 20 is cyclic.
   (b) The two permutations $(1\,2\,3\,4)$ and $(1\,2\,3\,4)(5\,6)$, both of order 4 in the symmetric group S_6, are conjugate.
   (c) The dihedral group D_4 of symmetries of the square with vertices $1, 2, 3, 4$ is a Sylow 2-subgroup of the symmetric group S_4.
   (d) In the cyclic group $G = \langle x \rangle$ of order 14, x^6 also has order 14.
   (e) $C_4 \times C_6 \cong C_{24}$.
   (f) A finite group can have 45 Sylow 11-subgroups.
   (g) In a commutative ring R with identity, if $a|b$ then $a \in (b)$.
   (h) A polynomial $f(x)$ of degree 3 in $\mathbf{Z}_5[x]$ has at most 3 zeros.
   (i) In $\mathbf{Z}(i)$, 2 is irreducible.

   [24 marks]

2. (a) If x, y and z are elements of order $2, 3, 4$ respectively in an abelian group G state the order of xy and the order of xz.
   (b) Give the torsion invariants and the rank of the abelian group G which is defined by:
   $$G = \langle a, b, c \mid a^4 b^3 c = a^{-8} b^{10} c^2 = a^{24} b^{-1} c = 1 \rangle\ .$$
   (c) Determine all abelian groups of order $252 = 2^2.3^2.7$, giving the prime power decomposition and torsion invariants for each group.

   [11 marks]

3. (a) Define the conjugacy class of a group G.
   (b) Let K be a conjugacy class of a group G. If $x \in K \cap Z(G)$ show that $K = \{x\}$.
   (c) Let a, g be elements of a finite group. Show that a and $g^{-1}ag$ have the same order.
   (d) If $\langle x \rangle$ has order 4 and $\langle x \rangle \triangleleft G$, give all possible conjugates of x and of x^2 in G. Hence show that $Z(G) \neq \langle 1 \rangle$.

   [12 marks]

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4. (a) State the Orbit-Stabilizer Theorem.

   (b) Let H be a subgroup of the finite group G with $G = Hx_1 \cup Hx_2 \cup \dots Hx_n$; that is, $H \cdot 1 = H = Hx_1, Hx_2, \dots, Hx_n$ are the n cosets of H in G. Let G act on the set $X = \{Hx_i \mid i = 1, \dots, n\}$ by $(Hx_i)^g = Hx_i g$.

      (i) Show that G has only one orbit $[H]$ on X.

      (ii) Let $S_G(H)$ denote the stabilizer of H in G. Use the Orbit-Stabilizer Theorem to determine $|S_G(H)|$.

      (iii) Hence determine $S_G(H)$.

[7 marks]

5. (a) State Sylow's Theorems on the existence, conjugacy and number of Sylow subgroups in a finite group G.

   (b) Let G be a group of order $p^a q^b$ where p, q are distinct primes and a, b are positive integers. If G has only one Sylow $p-$subgroup P and only one Sylow $q-$subgroup Q, explain why $G = P \times Q$.

   (c) Let G be a group of order 33. Use Sylow's Theorem and (b) to prove that G is cyclic.

[14 marks]

6. (a) Suppose ν_1, ν_2 denote the usual Euclidean norms for $\mathbf{Q}[x]$ and $\mathbf{Z}(i)$ respectively.

      (i) What are the functions ν_1, ν_2?

      (ii) Describe the elements $q \in \mathbf{Q}[x]$ with $\nu_1(q) = 0$.

      (iii) List the elements $z \in \mathbf{Z}(i)$ with $\nu_2(z) = 1$.

      (iv) Are $\mathbf{Q}[x], \mathbf{Z}(i)$ unique factorization domains? Give a reason for your answer.

   (b) Factor $p(x) = x^3 - x^2 + 2x - 2$ in $\mathbf{Z}_5[x]$ into irreducible factors. Use your factorisation to explain why $\mathbf{Z}_5[x]/(p(x))$ is *not* an integral domain.

   (c) (i) Define an *irreducible* element and a maximal ideal in a ring.

      (ii) Show that if R is a principal ideal domain then $M = (m)$ is a maximal ideal if m is irreducible.

      (iii) Using the integral domain $\mathbf{Z}[x]$, give an example of an irreducible element z for which (z) is not maximal.

[18 marks]

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7. Consider the integral domain $\mathbf{Z}(\sqrt{-5}) = \{a + b\sqrt{-5} \mid a, b \in \mathbf{Z}\}$.

   In this question you may use (without proof) the fact that that norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$ is multiplicative in $\mathbf{Z}(\sqrt{-5})$.

   (a) Prove that $\alpha \in \mathbf{Z}(\sqrt{-5})$ is a unit if and only if $N(\alpha) = 1$.

   (b) Find all units of $\mathbf{Z}(\sqrt{-5})$.

   (c) Show that if $N(\alpha) = 9$, then α is irreducible.

   (d) By considering the product $(1 + \sqrt{-5})(1 - \sqrt{-5})$, show that 3 is not prime in $\mathbf{Z}(\sqrt{-5})$.

   (e) Is $\mathbf{Z}(\sqrt{-5})$ a unique factorization domain? Justify your answer.

[14 marks]