Groups and Rings III 2009

Assignment 6

- Please hand up solutions to the starred questions by the 3.00 pm lecture on Friday 5th June.
- Please try the unstarred questions by the tutorial on Wednesday 3rd June at 9.00 at which they will be discussed.

1*. Let D be an integral domain. Show that

(a) $a \mid b$ if and only if $\langle a \rangle \supseteq \langle b \rangle$

(b) $\langle a \rangle = D$ if and only if a is a unit

(c) $\langle a \rangle = \langle b \rangle$ if and only if a and b are associates.

2. Let a and b be positive integers. We say that c is a common divisor of a and b if $c \mid a$ and $c \mid b$. We call the largest common divisor of a and b the greatest common divisor of a and b or $\gcd(a, b)$. Let $\langle a, b \rangle$ be the ideal generated by a and b. Using the fact that $\mathbb{Z}$ is a PID show that

$$\langle a, b \rangle = \langle \gcd(a, b) \rangle.$$

Deduce that there exist integers m and n such that $ma + nb = \gcd(a, b)$ and if c is a common divisor of a and b then $c \mid \gcd(a, b)$.

3*. Consider the ring of Gaussian Integers, $\mathbb{Z}(i) = \{a + bi \mid a, b \in \mathbb{Z}\}$ with Euclidean valuation $\delta(a + bi) = a^2 + b^2$.

(a) For $a = 1 - 2i$, $b = 3 - i$, find $q, r \in \mathbb{Z}(i)$ such that $a = bq + r$, with $\delta(r) < \delta(b)$, where δ is the Euclidean norm for $\mathbb{Z}(i)$.

(b) For each of 2 and 3 either show that they are irreducible or factorise them into products of irreducibles in $\mathbb{Z}(i)$.

4. Let G be the set of all units in a ring with identity. Show that G is a group with operation the ring multiplication. For the ring $\mathbb{Z}(\sqrt{2})$ show that the group of units is infinite.

5*. Consider the integral domain $D = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ with norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$. You may assume that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in D$.

(a) Prove that $\alpha \in D$ is a unit if and only if $N(\alpha) = 1$.

(b) Find all units of D.

(c) Show that if $N(\alpha) = 9$, then α is irreducible.

(d) By considering the product $(2 + \sqrt{-5})(2 - \sqrt{-5})$, show that 3 is not prime in D.

(e) Is D a unique factorization domain? Justify your answer. (Hint: In case we haven't got to it by the time you do this in a UFD primes are the same things as irreducibles.)

(Exam 2008)

6. Consider the integral domain $\mathbb{Z}(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$ with the norm $N(a + b\sqrt{10}) = a^2 - 10b^2$.

(a) Use the norm to describe the units of $\mathbb{Z}(\sqrt{10})$.

(b) By considering $a^2 \mod 10$ show that for any $x \in \mathbb{Z}(\sqrt{10})$ we have that $N(x) \mod 10$ can only be $0, 1, 4, 5, 6, 9$.

(c) Using (b) show that 2, 3, $4+\sqrt{10}$ and $4-\sqrt{10}$ are irreducible in $\mathbb{Z}(\sqrt{10})$.

(d) Is $\mathbb{Z}(\sqrt{10})$ a unique factorization domain ?

7*. Find all zeros, and hence factorise the following polynomials:

(a) $p(x) = x^3 - x^2 + 2x - 2$ in $\mathbb{Z}_3[x]$;

(b) $q(x) = x^4 - 4x^3 + x^2 - 4x$ in $\mathbb{Z}_5[x]$

8. Find all zeros, and hence factorise the following polynomials:

(a) $p(x) = x^3 + 2x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$;

(b) $q(x) = x^4 + 4$ in $\mathbb{Z}_5[x]$