Groups and Rings III 2009

Assignment 6

• Please hand up solutions to the starred questions by the 3.00 pm lecture on Friday 5th June.
• Please try the unstarred questions by the tutorial on Wednesday 3rd June at 9.00 at which they will be discussed.

1*. Let \( D \) be an integral domain. Show that

(a) \( a \mid b \) if and only if \( \langle a \rangle \supseteq \langle b \rangle \)
(b) \( \langle a \rangle = D \) if and only if \( a \) is a unit
(c) \( \langle a \rangle = \langle b \rangle \) if and only if \( a \) and \( b \) are associates.

2. Let \( a \) and \( b \) be positive integers. We say that \( c \) is a common divisor of \( a \) and \( b \) if \( c \mid a \) and \( c \mid b \). We call the largest common divisor of \( a \) and \( b \) the greatest common divisor of \( a \) and \( b \) or \( \gcd(a,b) \).

Let \( \langle a,b \rangle \) be the ideal generated by \( a \) and \( b \). Using the fact that \( \mathbb{Z} \) is a PID show that \( \langle a,b \rangle = \langle \gcd(a,b) \rangle \).

Deduce that there exist integers \( m \) and \( n \) such that \( ma + nb = \gcd(a,b) \) and if \( c \) is a common divisor of \( a \) and \( b \) then \( c \mid \gcd(a,b) \).

3*. Consider the ring of Gaussian Integers, \( \mathbb{Z}(i) = \{a+bi \mid a,b \in \mathbb{Z}\} \) with Euclidean valuation \( \delta(a+bi) = a^2 + b^2 \).

(a) For \( a = 1 - 2i, \ b = 3 - i \), find \( q,r \in \mathbb{Z}(i) \) such that \( a = bq + r \), with \( \delta(r) < \delta(b) \), where \( \delta \) is the Euclidean norm for \( \mathbb{Z}(i) \).
(b) For each of 2 and 3 either show that they are irreducible or factorise them into products of irreducibles in \( \mathbb{Z}(i) \).

4. Let \( G \) be the set of all units in a ring with identity. Show that \( G \) is a group with operation the ring multiplication. For the ring \( \mathbb{Z}(\sqrt{2}) \) show that the group of units is infinite.

5*. Consider the integral domain \( D = \{a+b\sqrt{-5} \mid a,b \in \mathbb{Z}\} \) with norm \( N(a+b\sqrt{-5}) = a^2 + 5b^2 \). You may assume that \( N(\alpha\beta) = N(\alpha)N(\beta) \) for all \( \alpha,\beta \in D \).

(a) Prove that \( \alpha \in D \) is a unit if and only if \( N(\alpha) = 1 \).
(b) Find all units of \( D \).
(c) Show that if \( N(\alpha) = 9 \), then \( \alpha \) is irreducible.
(d) By considering the product \( (2 + \sqrt{-5})(2 - \sqrt{-5}) \), show that \( 3 \) is not prime in \( D \).
(e) Is \( D \) a unique factorization domain? Justify your answer. (Hint: In case we haven’t got to it by the time you do this in a UFD primes are the same things as irreducibles.)

(Exam 2008)

6. Consider the integral domain \( \mathbb{Z}(\sqrt{10}) = \{a+b\sqrt{10} \mid a,b \in \mathbb{Z}\} \) with the norm \( N(a+b\sqrt{10}) = a^2 - 10b^2 \).

(a) Use the norm to describe the units of \( \mathbb{Z}(\sqrt{10}) \).
(b) By considering \( a^2 \mod 10 \) show that for any \( x \in \mathbb{Z}(\sqrt{10}) \) we have that \( N(x) \mod 10 \) can only be \( 0,1,4,5,6,9 \).
(c) Using (b) show that 2, 3, 4 + \sqrt{10} and 4 − \sqrt{10} are irreducible in \( \mathbb{Z}(\sqrt{10}) \).

(d) Is \( \mathbb{Z}(\sqrt{10}) \) a unique factorization domain?

7*. Find all zeros, and hence factorise the following polynomials:

(a) \( p(x) = x^3 - x^2 + 2x - 2 \) in \( \mathbb{Z}_3[x] \);
(b) \( q(x) = x^4 - 4x^3 + x^2 - 4x \) in \( \mathbb{Z}_5[x] \)

8. Find all zeros, and hence factorise the following polynomials:

(a) \( p(x) = x^3 + 2x^2 + 2x + 1 \) in \( \mathbb{Z}_7[x] \);
(b) \( q(x) = x^4 + 4 \) in \( \mathbb{Z}_5[x] \)