Groups and Rings III 2009

Assignment 4.

- Please hand up solutions to the starred questions by the 9.00am lecture on Wednesday 13th May. Either in the lecture or if earlier under the door of my office.
- Please try the unstarred questions by the tutorial on Wednesday 6th May at 9.00 at which they will be discussed.

1*. Find the torsion invariants and free rank of the abelian group

$$\langle a, b, c \mid a^3b^3c^6 = 1,\ a^{-3}b^9c^6 = 1,\ b^{-9}c^{-9} = 1\rangle$$

2. Find the torsion invariants and free rank of the abelian groups

(a) $\langle a, b, c \mid a^3b^3c^6 = 1,\ a^{-3}b^9c^6 = 1,\ b^{-9}c^{-9} = 1\rangle$

(b) $\langle a, b, c, d \mid a^3b^{-4}c^2d^{-4} = 1,\ a^3b^{-4}c^8d^8 = 1,\ a^{11}b^{-10}c^{18}d^4 = 1\rangle$.

3*. Recall that if v is a non-zero vector in $\mathbb{R}^n$ then there exists $v_1, \dots, v_{n-1}$ such that $v, v_1, \dots, v_{n-1}$ is a basis. Show that if v and w are non-zero vectors in $\mathbb{R}^n$ then there exists an invertible matrix A such that $Av = w$.

4*. Let the group G act on the set X.

(a) If $x \in X$ show that the stabiliser of G in x:

$$S_G(x) = \{g \in G \mid g \star x = x\}$$

is a subgroup of G.

(b) Recall that we call an action transitive if it has exactly one orbit. Show that an action is transitive if and only if for every $x, y \in X$ there is a G such that $g \star x = y$. In such a case show that

$$\{g \in G \mid g \star x = y\}$$

is a coset of $S_G(x)$.

5. Let X be a set and denote by S_X the set of all one to one and onto functions $\phi\colon X \to X$. S_X is a group under composition of functions.

(a) If X and Y are sets and $\rho\colon X \to Y$ is a one to one and onto function show that

$$\begin{array}{ccc} S_X & \to & S_Y \\ \phi & \mapsto & \rho \circ \phi \circ \rho^{-1} \end{array}$$

is an isomorphism of groups.

(b) Deduce that if X is a finite set then $S_X \simeq S_{|X|}$.

(c) Let $X = \{1, 2, \dots, n\}$ be acted on by S_n in the usual way. Use (c) to show that for any $i \in X$ we have $S_{S_n}(i) \simeq S_{n-1}$.

6*. Let G be a group and $S(G)$ be the set of all subgroups of G. The group G acts on $S(G)$ by conjugation:

$$g \star H = gHg^{-1}.$$

(a) Which subgroups lie in an orbit of length one?

(b) Which groups G have exactly *two* orbits?

(c) Which groups G have exactly *two orbits of length one*?

(d) Determine the orbits of S_3 on $S(S_3)$.

7. Let G be a group and $S(G)$ be the set of all subgroups of G. The group G acts on $S(G)$ by conjugation:

$$g \star H = gHg^{-1}.$$

(a) What is $S_G(H)$ for $H \in S(G)$?

(b) If H is a subgroup of G use the Orbit-Stabiliser Theorem to reprove the result from lectures that the number distinct cosets of H is $(G : N_G(H))$.

8. Let G be a group with $|G| = 504$. Show that G has to have subgroups of orders 7, 8 and 9.

9*. Let G be a group with $|G| = 1100$. Show that G has to have subgroups of orders 4, 11 and 25.

10. Let X be the set of all polynomials $f(x_1, \ldots, x_n)$ with integer coefficients. If $\sigma \in S_n$, define $\sigma \star f$ by

$$(\sigma \star f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

For example: $(1\ 2\ 3) \star (x_1 + 4x_3^2) = x_2 + 4x_1^2$. It can be shown that that this is an action of S_n on X.

Consider the polynomial $g(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

(a) Show that for any transposition $(k\ l)$ we have $(k\ l) * g = -g$.

(b) Deduce that the orbit of g is $\{g, -g\}$.

(c) Deduce that for all $\sigma \in S_n$, $\sigma * g = g \Leftrightarrow \sigma$ is the product of an even number of transpositions.

(d) Deduce that the stabilizer of g is A_n.

11*. The symmetric group S_4 acts on the set $X = \{(i, j) \mid 1 \leq i, j \leq 4\}$ with the following action:

$$g \star (i, j) = (g(i), g(j)) \quad \forall g \in S_4, (i, j) \in X.$$

For example, if $g = (1\ 2\ 3)$, then $g \star (2, 4) = (3, 4)$ and $g \star (1, 3) = (2, 1)$.

Show that with this action, S_4 has exactly *two* orbits on X, and give a representative of each orbit.

Verify the Orbit-Stabilizer Theorem for these two representatives.

12. Let p be a prime and consider $G = C_p$. Let n be a positive integer. A *bracelet* consists of p beads, each of which can be any of n different colours, placed in a circle. The group G acts on the set B of all bracelets by *rotation.*

(a) Show that the number of bracelets fixed by $g \in G$ is given by

$$B_g = \begin{cases} n^p, & \text{if } g = e \\ n, & \text{if } g \neq e. \end{cases}$$

(b) Hence find the number of essentially different bracelets. (Two bracelets are thought of as the same if one can be rotated into the other.) Calculate this number if $p = 17, n = 4$.