Groups and Rings III 2009

Assignment 3.

- Please hand up solutions to the starred questions during or before the 9.00 lecture on Tuesday 28th April.
- Please try the unstarred questions by the tutorial on Wednesday 8th April 9.00 at which they will be discussed.
- Please note I will be not be available to answer questions during the mid-semester break.
- Note that I use the notation $a \mid b$ to mean that a divides b.

1*. If C_m is a subgroup of C_n explain why it is a normal subgroup and show that quotient subgroup C_m/C_n is $C_{n/m}$.

2*. (a) Let C_{12} be generated by x and let $H = \langle x^4 \rangle$. List the elements of the factor group C_{12}/H. Find a generator of C_{12}/H and hence show that it is cyclic.

(b) Find all the composition series of C_{24}. Show that in each case the length of the composition series and the set of quotients is the same.

3. Let $G = G_1 \times G_2 \times \cdots \times G_k$ be a direct product of finite groups and let $g \in G$ with $g = (g_1, \ldots, g_k)$ where each $g_i \in G_i$. Let d have the property that $|g_i| \mid d$ for all $i = 1, \ldots, k$. Show that $g^d = e$.

4. Recall the fact that every natural number can be decomposed uniquely into a product of primes.

(a) Let a and b be natural numbers. By comparing the common primes in the prime decompositions of a and b show that
$$\operatorname{lcm}(a,b)\gcd(a,b) = ab$$
where $\operatorname{lcm}(a,b)$ is the least common multiple of a and b and $\gcd(a,b)$ is the greatest common divisor of a and b.

(b) Show that $C_m \times C_n \simeq C_{mn}$ if and only if $\gcd(m,n) = 1$.

(c) Using this result show that if $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i are distinct primes then
$$C_d \simeq C_{p_1^{a_1}} \times C_{p_2^{a_2}} \times \cdots \times C_{p_k^{a_k}}$$

5. (a) If n is a natural number and G any group show that any homomorphism $f\colon C_n \to G$ has cyclic image. (Hint: You may use Question 1 or argue directly.)

(b) If n and m are co-prime (that is $\gcd(m,n) = 1$) show that the only homomorphism $f\colon C_n \to C_m$ is given by $f(g) = e$ for all $g \in C_n$.

6*. Let H and K be groups and define a binary operation
$$H \times K \times H \times K \to H \times K$$
$$((h_1,k_1)(h_2,k_2)) \mapsto (h_1h_2, k_1k_2)$$

(a) Show that this binary operation makes $H \times K$ is a group.

(b) Show that $H_0 = \{(h,e) \mid h \in H\}$ is a subgroup of $H \times K$.

(c) Show that the map $\iota_H\colon H \to H \times K$ defined by $\iota_H(h) = (h,k)$ is a one-to-one homomorphism with image H_0.

(d) Show that the map $\pi_K\colon H \times K \to K$ defined by $\pi_K((h,k)) = k$ is an onto homomorphism with kernel H_0.

7. Let G be an abelian group.

(a) Show that $\mathrm{Tor}(G)$, the set of all elements of finite order, is a subgroup of G.

(b) Show that $G/\mathrm{Tor}(G)$ is a torsion-free group.

8*. Let $U = \{z \in \mathbb{C}^\times \mid |z| = 1\} < \mathbb{C}^\times$ and $\mathbb{R}_{>0} = \{x \in \mathbb{R}^\times \mid x > 0\}$. Show that:

(a) $\mathbb{C}^\times \simeq U \times \mathbb{R}_{>0}$. (Hint: polar decomposition)

(b) $\mathbb{R}^\times \simeq \mathbb{Z}_2 \times \mathbb{R}_{>0}$.

9. For each of the following groups find the torsion coefficients.

(a) $C_8 \times C_{30} \times C_{60}$

(b) $C_{12} \times C_{30} \times C_{150}$

10*. For each of the following groups find the torsion coefficients.

(a) $C_{30} \times C_{36} \times C_{80}$

(b) $C_{35} \times C_{55} \times C_{75}$

11. Find all abelian groups of order 504.

12*. Find all abelian groups of order 450.

[Additional questions can be found in Fraleigh.]