Groups and Rings III 2009

Assignment 2.

- Please hand up solutions to the starred questions by the 4.00 pm on Friday 27th March. Either in the lecture or if earlier under the door of my office.
- Please try the unstarred questions by the tutorial on Wednesday 25th at 9.00 at which they will be discussed.

1*. (a) If G is a finite group and $g \in G$ show that $g^{|G|} = e$.

(b) Prove Fermat's Little Theorem which says that if p is a prime and $0 < a < p$ then $a^{p-1} \equiv 1 \mod p$.

(c) Let $H \triangleleft G$ and $(G : H) = m$. Show that for any $g \in G$ we have $g^m \in H$.

2. In this question we determine the conjugacy classes of S_n.

(a) Let $a = (i_1 i_2 \cdots i_r)$ be a cycle and τ any permutation. Show that $\tau a \tau^{-1} = (\tau(i_1)\tau(i_2)\dots\tau(i_r))$. Use this to argue that any two cycles of the same length are conjugate.

(b) Define the cycle structure of a permutation π to be the unordered lengths of the cycles in a decomposition of π into disjoint cycles. Deduce that any two permutations are conjugate if they have the same cycle structure.

(c) Show that any two permutations which are conjugate have the same cycle structure.

(d) Recalculate the conjugacy classes of S_3 which we did in class using these results.

3*. Determine the conjugacy classes of S_4. You may use the result from Question 1. Pick an element π in each class and determine $C_{S_4}(\pi)$.

4. . Consider the quaternion group $\mathbb{H}$.

(a) Decompose $\mathbb{H}$ into conjugacy classes.

(b) Calculate $C_G(x)$ for each $x \in \mathbb{H}$.

(c) Calculate $Z(\mathbb{H})$.

5*. (a) If G is a group show that $Z(G)$, the centre of G, is a subgroup of G.

(b) Show that if $xy \in Z(G)$ then $xy = yx$.

(c) Show that $Z(GL(n, \mathbb{C})) \simeq \mathbb{C}^\times$ and $Z(SL(n, \mathbb{C})) \simeq U_n$. (Hint: Assume X is in the centre and consider the equation $EX = XE$ where E is an elementary matrix as in Mathematics I. Try E so that EX is X with a row multiplied by a constant $a \in \mathbb{C}^\times$ and then try an E that does a row swap. Remember that when you multiply on the left by E it performs elementary column operations.)

6. Consider the group D_4 of symmetries of the square whose vertices are labelled anti-clockwise. Regard it as a subgroup of S_4 by the permutations of the vertices so that

$$D = \{1, (1234), (13)(24), (1432), (14)(23), (12)(43), (13), (24)\}.$$

(a) Partition D_4 into conjugacy classes $[x_1], [x_2], \dots, [x_r]$.

(b) Calculate $|C_{D_4}(x_i)|$ for each x_i and hence find the group $C_G(x_i)$ for each x_i.

7*. (a) If $H < G$ prove that $N_G(H)$ is a subgroup of G containing H.

(b) In S_3 find all the conjugates of $H = \{1, (12)\}$.

(c) Find the normaliser of H in S_3. You can use results from class. Verify the formula for the number of conjugacy classes we proved in class.

8. (a) If H and K are subgroups of G with $H \triangleleft G$ show that

$$HK = \{hk \mid h \in Hk \in K\}$$

is a subgroup of G.

(b) Let H and K be subgroups of a group G. Show that HK is a subgroup of G if an only if $HK = KH$.

9. Let G be a group with $N \triangleleft G$ and $N \neq \langle e \rangle$.

(a) Prove that N is a union of conjugacy classes.

(b) If G is a p-group show that $Z(G) \cap N = \langle e \rangle$. (Hint: Use the same idea that was used in class to show that $Z(G) \neq \langle e \rangle$ for p-groups.)

10*. (a) If $g \in G$ define a function $\mathrm{Ad}_g \colon G \to G$ by $\mathrm{Ad}_g(x) = gxg^{-1}$. Show that Ad_g is a group homomorphism.

(b) If H is a subgroup of G and $g \in G$ show that gHg^{-1} is a subgroup of G which is isomorphic to H.

(c) If x and y are conjugate in G show that $C_G(x)$ and $C_G(y)$ are isomorphic.

11. Let $f \colon G \to H$ be a homomorphism of groups. If $K \subset G$ define $f(K) = \{f(k) \mid k \in K\} \subset H$ and if $L \subset H$ define $f^{-1}(L) = \{g \in G \mid f(g) \in L\} \subset G$.

(a) If $K < G$ show that $f(K) < H$.

(b) If $L < H$ show that $f^{-1}(L) < G$.

(c) If $K \triangleleft G$ and f is onto show that $f(K) \triangleleft H$.

(d) If $L \triangleleft H$ show that $f^{-1}(L) \triangleleft G$.

12*. For $a \neq 0$ and b in $\mathbb{Z}_7$ define a function

$$F_{a,b} \colon \mathbb{Z}_7 \to \mathbb{Z}_7$$

by $F_{a,b}(x) = ax + b$. The set of all these functions forms a group G under composition. (You may assume this.)

(a) Determine $|G|$ and calculate $F_{a,b} \circ F_{c,d}(x) = F_{ab}(F_{c,d}(x))$. Use this to find e, f such that $F_{a,b} \circ F_{c,d} = F_{e,f}$.

(b) Show that G is non-abelian.

(c) Show that the mapping $f \colon G \to \mathbb{Z}_7^\times$ given by $f(F_{a,b}) = a$ is a homomorphism. Find $\ker(f)$. Write down a normal subgroup of G.

[Additional questions can be found in Fraleigh.]