Groups and Rings III 2009

Assignment 1.

Please hand up solutions to the starred questions by the 4.00 pm on Friday 13th March. Either in the lecture or if earlier under the door of my office.

1*. Consider the group $GL(2,\mathbb{R})$ of 2×2 invertible matrices with operation being matrix multiplication. Define two subsets by

$$G = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \,\middle|\, \alpha, \beta, \delta \in \mathbb{R}, \alpha\delta \neq 0 \right\} \quad \text{and} \quad H = \left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \,\middle|\, \beta \in \mathbb{R} \right\}.$$

(a) Show that G and H are subgroups of $GL(2,\mathbb{R})$.

(b) Show that H is abelian. Is G abelian ? Justify your answer.

(c) Show that $H \lhd G$? Is $H \lhd GL(2,\mathbb{R})$? Justify your answer.

(d) Show that H is isomorphic to $(\mathbb{R}, +)$.

2. Consider D_3 the group of symmmetries of an equilatorial triangle. Write down all the symmetries and their orders, classifying them as rotations or reflections. Label the vertices and show that every permutation in S_3 arises as a symmetry of the equilatorial triangle.

3*. Let G be a group.

(a) Show that $\emptyset \neq H \subset G$ is a subgroup if and only if for all $x, y \in H$ we have $xy^{-1} \in H$.

(b) Show that if H and K are subgroups of G then $H \cap K$ is a subgroup of G.

4. Let G be a group. Show that for any $a, b, c \in G$ we have

(a) (i) $|a^{-1}| = |a|$ (ii) $|b^{-1}ab| = |a|$ (iii) $|ab| = |ba|$ (iv) $|abc| = |bca| = |cab|$

In each case also show that either both sides are finite or both sides are infinite.

(b) Consider the permutations $a = (123)$, $b = (14)$ and $c = (24)$. Find $|abc|$ and $|bac|$ and compare. Does this contradict the first part of the question?

5*. Let G be a group.

(a) Show that if $|x| = 2$ for all $x \neq e$ in G then G is abelian. (Hint: Consider $(ab)^2$.)

(b) Show that if G has *exactly* one element x of order 2 then x commutes with everything in G. (Hint: Consider the order of yxy^{-1} for $y \in G$).

6*. Write down the multiplication tables of V_4 the group of symmetries of a rectangle and U_4 the group of fourth roots of unity. Are they isomorphic ? Justify your answer.

7. Let $D = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \,\middle|\, \alpha, \beta \in GF(5), \alpha\beta \neq 0 \right\}$; i.e. D is the subgroup of diagonal matrices of $GL(2,5)$.

(a) Write down $|D|$ and show that D is abelian.

(b) Is D cyclic? If D is cyclic find a generator; if not, find the smallest number of matrices A, B, C,... such that $D = \langle A, B, C, \ldots\rangle$.

(c) Determine $|D \cap SL(2,5)|$.

8*. Recall that $\mathbb{R}^\times = \mathbb{R} - \{0\}$ and let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \subset \mathbb{R}^\times$.

(a) Show that $(\mathbb{R}^+, \times)$ is a subgroup of $(\mathbb{R}^\times, \times)$.

(b) Show that $(\mathbb{R}^+, \times)$ is isomorphic to the group $(\mathbb{R}, +)$. (Hint: Think about log and exp.)

(c) Explain why $(\mathbb{R}, +)$ is not isomorphic to $(\mathbb{R}^\times, \times)$. (Hint: Elements of order two.)

(d) Is $(\mathbb{Q}, +)$ cyclic ?

9. In the cyclic group $\mathbb{Z}_{27}$ determine the orders of 6 and 9. Find all the (cyclic) subgroups and determine the subgroup lattice of $\mathbb{Z}_{27}$.

10*. In the cyclic group $C_{36} = \langle x \rangle$ determine the orders of x^{17} and x^{12}. Find all the (cyclic) subgroups and determine the subgroup lattice of C_{36}.

11. Consider the following permutations in S_6, the symmetric group on 6 letters.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}.$$

(a) Write each of σ and τ as a product of disjoint cycles and find their orders.

(b) Write down σ^{-1} and find $\sigma^{-1}\tau$.

(c) Is $\sigma^{-1}\tau$ an even or odd permutation?

12*. Consider the following permutations in S_5, the symmetric group on 5 letters.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}.$$

(a) Write each of σ and τ as a product of disjoint cycles and find their orders.

(b) Write down σ^{-1} and find $\sigma^{-1}\tau$.

(c) Is $\sigma^{-1}\tau$ an even or odd permutation?

[Additional questions can be found in Fraleigh.]