Differential Geometry

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1. Introduction

These notes have been used for teaching the Honours topic Differential Geometry IV at the University of Adelaide over many years. I have listed some of the sources in the References but I have probably forgotten others. My thanks to the many students who gave me corrections over the years. Particularly to Michael Hallam for an extensive collection of corrections in 2016.

The presentation differs in the degree of detail with the last part on de Rham cohomology being only a sketch.

2. Sets, relations and functions

We need to know some basic things about sets, relations and functions.

Definition 2.1. If A is a set a *partition* of A is a collection of subsets $\{A_\alpha\}_{\alpha\in I}$ indexed by elements of some set I with the property that if $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$ and $A = \bigcup_{\alpha\in I} A_\alpha$.

Definition 2.2. If A and B are sets then a *relation between A and B* is a subset $R \subseteq A \times A$. Often we write it as $a \simeq b$ if $(a, b) \in R$. If $A = B$ we call a relation between A and A a *relation on A*.

Definition 2.3. We say a relation on A is an *equivalence relation* if

(1) for all $a \in A$ we have $a \simeq a$ (reflexive)
(2) for all $a_1, a_2 \in A$ if $a_1 \simeq a_2$ then $a_2 \simeq a_1$ (symmetric), and
(3) for all $a_1, a_2, a_3 \in A$ if $a_1 \simeq a_2$ and $a_2 \simeq a_3$ then $a_1 \simeq a_3$ (transitive).

Definition 2.4. If $\simeq$ is an equivalence relation on A then we define the equivalence class of $a \in A$ to be

$$[a] = \{b \in A \mid b \simeq a\} \subseteq A$$

Exercise 2.1. Show that the equivalence classes of an equivalence relation on A partition A.

Definition 2.5. Let $\simeq$ be an equivalence relation on a set A. Denote by $A/\simeq$ the set of all equivalence classes equipped with the surjective map $\pi\colon A \to A/\simeq$ defined by $\pi(a) = [a]$. We call $A/\simeq$ the *quotient of A by the equivalence relation*.

Exercise 2.2. On $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ define a relation by $(a, b) \simeq (c, d)$ if $ad = bc$. Show that quotient of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ by this equivalence relation can be identified with $\mathbb{Q}$ the rational numbers.

Exercise 2.3. Let V be a vector space. Define an equivalence relation $\simeq$ on $V - \{0\}$ by $v \simeq w$ if there is some non-zero number λ such that $v = \lambda w$. Show that $\simeq$ is an equivalence relation. We can identify the quotient of $V - \{0\}$ with the set of all lines in V through 0 (i.e one-dimensional subspaces of V) and we call it the *projective space* of V and denote it by $P(V)$.

Notice that we can do this for either real or complex vector spaces and we get real projective space $\mathbb{R}P_n = P(\mathbb{R}^{n+1})$ and complex projective space $\mathbb{C}P_n = P(\mathbb{C}^{n+1})$.

A *function* $f\colon A \to B$ is a relation $R \subseteq A \times B$ between A and B of a special kind. We require that for every $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in R$. We write $b = f(a)$ and note that R is what we usually call the graph of f.

Recall that if $f\colon A \to B$ and $C \subseteq A$ then $f|_C : C \to B$ is the *restriction of f to C* defined by $f|_C(c) = f(c)$ for all $c \in C$. We also need to think about composition of functions. Strictly speaking we can only compose $f\colon A \to B$ and $g\colon C \to D$ if $B = C$ and then we define the *composition of g and f* written $g \circ f\colon A \to D$ by $(g \circ f)(a) = g(f(a))$. However we will often be in the situation where $B \subseteq C$ and we will just write $g \circ f$ for what is really $g|_B \circ f$.

Recall that if $f\colon A \to B$ and $D \subseteq B$ then we write

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\} \subseteq A$$

for the *pre-image* of D under f. There is no implication or assumption in this notation that f admits an inverse function. Likewise if $C \subseteq A$ we write

$$f(C) = \{f(c) \mid c \in C\} \subseteq B$$

for the *image* of C under f.

Exercise 2.4. Let $f\colon A \to B$. If $C \subseteq A$ show that $C \subseteq f^{-1}(f(C))$ and if $D \subseteq B$ then $f(f^{-1}(D)) = D$. Show by an example that we cannot expect to improve the first inclusion to an equality.

Exercise 2.5. If $f\colon A \to B$ is a function define $a_1 \simeq a_2$ if $f(a_1) = f(a_2)$. Show that $\simeq$ is an equivalence relation and that $[a] = f^{-1}(f(a))$ for all $a \in A$.

Generally I will follow standard mathematical practice and call $1-1$ functions *injective*, onto functions *surjective* and those which are both *bijective*.

3. Co-ordinate independent calculus.

3.1. **Introduction.** In this section we review some elementary constructions from multivariable calculus. We will formulate them in a way that makes their dependence on co-ordinates manifest. This will make the transition to calculus on manifolds simpler.

3.2. **Smooth functions.** Recall that if $f\colon U \to \mathbb{R}$ is a function defined on an open subset U of $\mathbb{R}$ then we say that f is *differentiable* at $a \in U$ if the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If the limit exists we call it the *derivative* of f at a and denote it by any of

$$df(a), \quad d_a f, \quad \text{or} \quad f'(a).$$

If f is differentiable at any x in U we just say that f is differentiable or differentiable on U.

There are two natural ways of generalising differentiation to higher dimensions. First there is the general definition of differentiability.

Definition 3.1. Let U be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$. We say that f is *differentiable at* $a \in U$ if there is a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0. \tag{3.1}$$

If f is differentiable at a it is not hard to show that the L in the definition is unique and we call it the *derivative* of f at a and denote it by $df(a)\colon \mathbb{R}^n \to \mathbb{R}^m$. We will give a formula for it in a moment. All the usual results about differentiability hold in particular differentiable functions are continuous.

The second direction we could head is that of partial derivatives. If $U \subseteq \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}$ we can define partial derivatives by varying only one of the co-ordinates and hence reducing to the one-variable case. If e^i is the element of $\mathbb{R}^n$ with a 1 in the ith position and 0's elsewhere we define a curve by

$$\gamma_i(t) = a + te_i.$$

The ith partial derivative of f at a is then defined by

$$\partial_i f(a) = (f \circ \gamma_i)'(0) = \frac{\partial f}{\partial x^i}(a).$$

Notice that we need to be a little careful here with domains of functions. We really have $\bar{\gamma}_i\colon \mathbb{R} \to \mathbb{R}^n$ defined by $\bar{\gamma}(t) = a + te_i$ which is continuous, and in fact differentiable of any order. So $V = \bar{\gamma}_i^{-1}(U)$ is open in $\mathbb{R}$ and the function $\gamma_i\colon V \to \mathbb{R}^n$ is the restriction of $\bar{\gamma}_i$ to V. Alternatively we could observe that V is an open subset of $\mathbb{R}$ containing 0 (because $\bar{\gamma}_i(0) = a \in U$) so there is an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq V$ and then restrict $\bar{\gamma}_i$ to $(-\epsilon, \epsilon)$ to get $\gamma_i\colon (-\epsilon, \epsilon) \to \mathbb{R}^n$. Generally, in future, we will assert the existence of ϵ without further comment.

If $f\colon U \to \mathbb{R}$ has continuous partial derivatives of all orders up to and including k we say that it is C^k and denote the set of all such functions by $C^k(U, \mathbb{R})$ or just $C^k(U)$. If f is in $C^k(U)$ for all k we say it is *smooth* and we denote the set of smooth functions by $C^\infty(U, \mathbb{R})$ or $C^\infty(U)$.

The basic result relating these two points of view is the following proposition.

Proposition 3.2 (See for example [7])**.** *If $f \in C^1(U)$ then f is differentiable at every $a \in U$.*

If $f\colon U \to \mathbb{R}^m$ then at any $x \in U$ we have $f(x) = (f^1(x), \dots, f^m(x))$ which defines m functions $f^i\colon U \to \mathbb{R}$ called the *component functions* or *components* of f. We have the fundamental results in the following propositions.

Proposition 3.3 (See for example [7])**.** *Let $f\colon U \to \mathbb{R}^m$. Then f is differentiable at $a \in U$ if and only if each f^i is differentiable at $a \in U$ for $i = 1, \dots, m$. Moreover $df(a)(v) = (df^1(a)(v), \dots, df^m(a)(v)) \in \mathbb{R}^m$.*

and

Proposition 3.4 (See for example [7])**.** *Let $f\colon U \to \mathbb{R}^m$ be differentiable at a. Then the linear map $L = df(a)$ in the limit (3.1) is given by*

$$df(a)(v)^t = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(a) & \cdots & \frac{\partial f^1}{\partial x^n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(a) & \cdots & \frac{\partial f^m}{\partial x^n}(a) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

If $f\colon U \to \mathbb{R}^m$ we define f to be C^k if each component f^i of f is C^k for $i = 1, \dots, m$. We denote by $C^k(U, \mathbb{R}^m)$ the set of all C^k functions on U.

3.3. **Derivatives as linear operators.** Because partial derivatives are co-ordinate dependent they are not a particularly useful way of thinking about derivatives if we want to move to a co-ordinate independent setting such as differentiable manifolds. It is more useful to think of the derivative of a function $f\colon U \to \mathbb{R}$ at x as a *linear* map

$$df(x)\colon \mathbb{R}^n \to \mathbb{R}$$

defined by

$$df(x)(v) = \frac{d}{dt}(t \mapsto f(x + tv))\bigg|_{t=0}.$$

We think of this as the rate of change of f at x in the direction of v. For smooth functions $df(x)$ is linear. Note that $df(x)$ is akin to the notion of a directional derivative but we do not require that v is of unit length. We can recover the partial derivatives from this definition by applying the linear operator $df(x)$ to the vector e^i. The result, $df(x)(e^i)$, is just the ith partial derivative of f at x.

Similarly if $f\colon U \to \mathbb{R}^m$ then we define a linear map

$$df(x)\colon \mathbb{R}^n \to \mathbb{R}^m$$

by

$$df(x)(v) = \frac{d}{dt}(t \mapsto f(x + tv))\bigg|_{t=0}.$$

As a linear map we can expand $df(x)$ in a basis and we recover the Jacobian matrix of partial derivatives by letting

$$df(x)(e_j) = \sum_{i=1}^m \partial_j f^i(x) e_i$$

for $j = 1, \dots, n$.

3.4. **The chain rule.** Fundamental to many of the constructions we want to consider in the following sections is the chain rule:

Theorem 3.5 (Chain Rule. See for example [7])**.** *Let $U \subseteq \mathbb{R}^n$ be open, $f\colon U \to \mathbb{R}^m$, $V \subseteq \mathbb{R}^m$ be open and $g\colon V \to \mathbb{R}^k$ with $f(U) \subseteq V$. Let $x \in U$. If f and g are smooth so also is $g \circ f\colon U \to \mathbb{R}^k$ and*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

The composition on the right hand side is the composition of linear operators. In particular if we expand both sides in terms of the standard basis of $\mathbb{R}^n$ then we have

$$\partial_j (g \circ f)^i(x) = \sum_{l=1}^m \partial_l g^i(f(x)) \circ \partial_j f^l(x)$$

An important part of the chain rule is the fact that the composition of smooth functions is also smooth. A partial converse of this result will be important in the sequel.

Lemma 3.6. *Let U be an open subset of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^m$. A function $\phi\colon U \to V$ is smooth if and only if for every smooth function $f\colon V \to \mathbb{R}$ the composite $f \circ \phi\colon U \to \mathbb{R}$ is smooth.*

Proof. If ϕ is smooth then the result follows via the chain rule. If the result is true then take f to be the restriction to V of each of the co-ordinate functions $x \mapsto x^i$. The co-ordinate functions are smooth so their composition with ϕ which is ϕ^i is smooth. But if each component ϕ^i is smooth so also is ϕ. ∎

3.5. **Diffeomorphisms and the inverse function theorem.** A function $f\colon U \to V$ where U and V are open subsets of $\mathbb{R}^n$ is called a (smooth) *diffeomorphism* if it is smooth, invertible and has smooth inverse $f^{-1}\colon V \to U$. If f is a diffeomorphism $f^{-1} \circ f = 1_U$ so it follows from the chain rule that at any point $x \in U$

$$1_{\mathbb{R}^n} = d(1_U)(x) = d(f^{-1} \circ f)(x) = (df^{-1})(f(x)) \circ df(x)$$

so that $(df(x))^{-1} = df^{-1}(f(x))$. That is, the inverse of the linear map $df(x)$ is the linear map $df^{-1}(f(x))$. Notice that this means that a diffeomorphism necessarily goes from an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^m$ where $n = m$ so we have lost nothing by putting that in the definition.

Example 3.1. It is worth noticing that it is not enough for f to be differentiable and have an inverse. For example $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is not a diffeomorphism. Note that $df(0) = 0$ so it cannot have inverse f^{-1} as if it did it would have to satisfy $d(f^{-1})(0) = 0^{-1}$. This is a complicated way of saying that the cube root function is not differentiable at 0!

It is also useful to have the notion of a *local diffeomorphism*. We say that $f\colon U \to \mathbb{R}^n$ is a local diffeomorphism at $a \in U$ if there is an open subset $V \subseteq U$ of containing x such that $f(V)$ is open and $f\colon V \to f(V)$ is a diffeomorphism.

With this notion we have the important inverse function theorem:

Theorem 3.7 (Inverse Function Theorem)**.** *Let U be an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ be a smooth function such that $df(a)$ is invertible at $a \in U$. Then f is a local diffeomorphism at a and $d(f^{-1})(f(a)) = (df(a))^{-1}$.*

Proof. Unfortunately proving this will take us too far afield. The proof uses the contraction mapping theorem and can be found for example in [7]. ∎

The Lemma proved in the previous section also gives us a characterisation of diffeomorphisms:

Lemma 3.8. *Let U and V be open subsets of $\mathbb{R}^n$. A bijection $\phi\colon U \to V$ is a diffeomorphism if and only if for every function $f\colon V \to \mathbb{R}$ we have that f is smooth if and only if $f \circ \phi\colon U \to \mathbb{R}$ is smooth.*

Proof. We just apply Lemma 3.6 to ϕ and ϕ^{-1}. ∎

4. Differentiable manifolds

4.1. **Co-ordinate charts.** Manifolds are sets on which we can define co-ordinates in such a way that we can do calculus. In general we don't expect to be able to define co-ordinates on all of a manifold any more than we can cover the whole of the surface of the earth with a single map. First we define:

Definition 4.1 (Co-ordinate charts)**.** A *co-ordinate chart* on a set M is a pair (U, ψ) where $U \subseteq M$ and $\psi\colon U \to \mathbb{R}^n$ is a function which is a bijection onto its image $\psi(U)$ which is open in $\mathbb{R}^n$.

If (U, ψ) is a co-ordinate chart we call U the *domain* of the co-ordinate chart and ψ the *co-ordinates*. Notice that we do not say that U is open in M because M is not a topological space yet; it is just a set.

Example 4.1. Let $1_{\mathbb{R}^n}\colon \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. That is $1_{\mathbb{R}^n}(x) = (x^1, \dots, x^n)$. Then $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ is a co-ordinate chart on $\mathbb{R}^n$. We usually call these the *standard*, *usual* or *natural* co-ordinates.

Example 4.2. Let U be any open subset of $\mathbb{R}^n$ and

$$\iota\colon U \to \mathbb{R}^n$$

the inclusion map defined by $\iota(x) = x$. Then clearly $\iota(U) = U$ which is open and ι is a bijection onto $\iota(U) = U$ so that (U, ι) is a co-ordinate chart on $\mathbb{R}^n$.

Example 4.3. Let V be a finite dimensional vector space. Choose a basis $v^1, \dots, v^n$ for V and define $\psi\colon V \to \mathbb{R}^n$ by

$$u = \sum_{i=1}^{n} \psi^i(u) v^i.$$

Then ψ is a bijection, in fact a linear isomorphism. Indeed every linear isomorphism arises in this way as if $\phi\colon V \to \mathbb{R}^n$ is a linear isomorphism we can take $w^i = \phi^{-1}(e^i)$ where e^i is the vector with a 1 in the ith place and zeros everywhere else. We leave it as an exercise to show that for every $u \in V$

$$u = \sum_{i=1}^{n} \phi^i(u) w^i.$$

Example 4.4. Let

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \leq 0\}$$

and define polar co-ordinates

$$(r, \theta)\colon U \to (0, \infty) \times (-\pi, \pi) \subseteq \mathbb{R}^2$$

as follows. We define $r\colon U \to (0, \infty)$ by $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta\colon U \to (-\pi, \pi)$ by the requirement that $x = r(x, y)\cos(\theta(x, y))$ and $y = r(x, y)\sin(\theta(x, y))$. Clearly (r, θ) is a bijection on the given domain and range.

Example 4.5. Let S^2 be the set of all points in $\mathbb{R}^3$ of length one. Let

$$U_0 = S^2 - \{(0, 0, 1)\} \subseteq S^2.$$

We can define co-ordinates on U_0 by *stereographic projection* from the point $(0, 0, 1)$ onto the X-Y plane. That is if $p = (x, y, z) \in U_0$ it has co-ordinates $\psi_0(p) = (\psi_0^1(p), \psi_0^2(p))$ defined uniquely by the requirement that the line through $(0, 0, 1)$ and p intersects the X-Y plane at $(\psi_0^1(p), \psi_0^2(p), 0)$. So we must have

$$(x - 0, y - 0, z - 1) = t(\psi_0^1(p) - 0, \psi_0^2(p) - 0, 0 - 1)$$

for some $t \in \mathbb{R}$ and hence

$$\psi_0^1(x, y, z) = \frac{x}{1 - z} \qquad \text{and} \qquad \psi_0^2(x, y, z) = \frac{y}{1 - z}.$$

Exercise 4.1. Let $S^n \subset \mathbb{R}^{n+1}$ denote the n-sphere of all points of length 1. Generalise the case of stereographic projection on the two-sphere to define co-ordinates on the n-sphere.

In general a manifold will have lots of co-ordinates. We no more expect a manifold to come with a given set of co-ordinates than we expect an abstract vector space to come with a given basis. However not all co-ordinate charts will do. We want them to be able to fit together in some compatible way. The motivation for our definition comes from the desire to define differentiable (in fact smooth) functions on a manifold. Indeed we can regard co-ordinates as a device to decide which, of the many functions on M, are going to be smooth. Let (U, ψ) be a co-ordinate chart and let $f\colon U \to \mathbb{R}$ be a function. Then as U is just a set it makes no sense to ask that f be smooth. However we can ask that f be smooth with respect to the co-ordinates. That is we consider

$$f \circ \psi^{-1}\colon \psi(U) \to \mathbb{R}.$$

Now $f \circ \psi^{-1}$ is a function defined on an open $U \subseteq \mathbb{R}^n$, namely $\psi(U)$ and we know what it means for such a function to be smooth. Consider now what happens when we change co-ordinates to some other co-ordinate chart say (V, χ), for convenience assuming that $V = U$. Then it is possible that $f \circ \psi^{-1}$ is smooth but $f \circ \chi^{-1}$ is not. To compare them we write

$$f \circ \psi^{-1} = f \circ \chi^{-1} \circ (\chi \circ \psi^{-1})$$

where

$$\chi \circ \psi^{-1}\colon \psi(U) \to \chi(V)$$

is a bijection between open subsets of $\mathbb{R}^n$. Then a sufficient condition for $f \circ \psi^{-1}$ to be smooth if $f \circ \chi^{-1}$ is is that $\chi \circ \psi^{-1}$ is smooth. As we want this to work both ways we also require that $\psi \circ \chi^{-1}$ be smooth. In other words we require that $\chi \circ \psi^{-1}$ is a diffeomorphism. If we want this to be true for any f then we have already seen in Lemma 3.6 that this becomes a necessary condition.

In practice we may not be able to find charts (U, ψ) and (V, χ) with $U = V$ so in the definition we need to allow for this.

Definition 4.2 (Compatibility of charts)**.** A pair of charts (U, ψ) and (V, χ) are called *compatible* if the sets $\psi(U \cap V)$ and $\chi(U \cap V)$ are open and the map

$$\chi \circ \psi^{-1}\big|_{\psi(U\cap V)} : \psi(U \cap V) \to \chi(U \cap V)$$

is a diffeomorphism.

Note that we need to restrict the map ψ^{-1} to the set $\psi(U \cap V)$ so that it can be composed with χ. In general just writing $\chi \circ \psi^{-1}$ will not make sense. Moreover the restriction comes before the composition so it is $\chi \circ \Big(\psi^{-1}\big|_{\psi(U\cap V)}\Big)$.

Example 4.6. If $U \subseteq \mathbb{R}^2$ is the set in example 4.4 on which polar co-ordinates are defined then it has two co-ordinate charts defined on it: $(U, (r, \theta))$ and (U, ι), the polar co-ordinates and the inclusion. Notice that $U \cap U = U$ so that $\iota(U \cap U)$ and $(r, \theta)(U \cap U)$ are open by assumption.

If we calculate the composition

$$\iota \circ (r, \theta)^{-1} \colon (0, \infty) \times (-\pi, \pi) \to U$$

we obtain

$$\iota \circ (r, \theta)^{-1}(s, \phi) = (s \cos(\phi), s \sin(\phi))$$

which is a diffeomorphism. Hence $(U, (r, \theta))$ and (U, ι) are compatible.

Example 4.7. Let V be a vector space and $v^1, \dots, v^n$ and $w^1, \dots, w^n$ bases defining co-ordinates ψ and ϕ by

$$v = \sum_{i=1}^{n} \psi^i(v) v^i = \sum_{i=1}^{n} \phi^i(v) w^i.$$

Both ϕ and ψ are onto so that $\psi(V \cap V) = \mathbb{R}^n$ is certainly open in $\mathbb{R}^n$ and likewise for ϕ. If we define a matrix X^i_j by

$$v^i = \sum_{j=1}^{n} X^i_j w^j$$

for all i then

$$\sum_{i,j=1}^{n} \psi^i(v) X^i_j w^j = \sum_{j=1}^{n} \phi^j(v) w^j$$

so that

$$\phi^j(v) = \sum_{i=1}^{n} X^i_j \psi^i(v).$$

Another way of calculating this result is to observe that

$$\phi \colon V \to \mathbb{R}^n$$

and

$$\psi \colon V \to \mathbb{R}^n$$

are linear isomorphisms so that

$$\phi \circ \psi^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$$

is the linear isomorphism with matrix X^i_j. Being linear $\phi \circ \psi^{-1}$ is certainly smooth so that (V, ϕ) and (V, ψ) are compatible.

Example 4.8. If we consider again the example of S^2 we had defined a co-ordinate chart (U_0, ψ_0) by

$$U_0 = S^2 - \{(0, 0, 1)\}$$

and

$$\psi_0(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

If we stereographically project from the point $(0, 0, -1)$ then we get co-ordinates

$$\psi_1(x, y, z) = \left(\frac{x}{1 + z}, \frac{y}{1 + z} \right).$$

defined on

$$U_1 = S^2 - \{(0,0,-1)\}$$

We want to check that these are compatible. Note first that both $\psi_0(U_0 \cap U_1)$ and $\psi_1(U_0 \cap U_1)$ are equal to $\mathbb{R}^2 - \{(0,0)\}$ which is open in $\mathbb{R}^2$. Then an easy calculation shows that

$$\psi_0 \circ \psi_1^{-1}\big|_{\mathbb{R}^2 - \{(0,0,0)\}}(x^1, x^2) = \left(\frac{x^1}{(x^1)^2 + (x^2)^2}, \frac{x^2}{(x^1)^2 + (x^2)^2} \right)$$

which is a diffeomorphism from $\mathbb{R}^2 - \{(0,0)\}$ to $\mathbb{R}^2 - \{(0,0)\}$. Similarly for $\psi_1 \circ \psi_0^{-1}\big|_{\mathbb{R}^2 - \{(0,0)\}}$.

To make M into a manifold we need to be able to cover it with compatible co-ordinate charts.

Definition 4.3 (Atlas)**.** An *atlas* for a set M is a collection $\mathcal{A} = \{(U_\alpha, \psi_\alpha) \mid \alpha \in I\}$ of co-ordinate charts such that:

(i) for any α and β in I, (U_α, ψ_α) and (U_β, ψ_β) are compatible and;

(ii) $M = \cup_{\alpha \in I} U_\alpha$.

Then we have

Definition 4.4 (Manifold)**.** A *manifold* is a set M with an atlas $\mathcal{A}$. We call the choice of an atlas $\mathcal{A}$ for a set M a choice of *differentiable structure* for M.

Example 4.9. If there is a co-ordinate chart with domain all of M then this, by itself defines an atlas and makes M a manifold. For example $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ makes $\mathbb{R}^n$ a manifold and if U is open in $\mathbb{R}^n$ then (U, ι) makes U a manifold.

Example 4.10. If V is a vector space then any linear isomorphism from V to $\mathbb{R}^n$ makes V a manifold. The vector space V has other atlases such as the atlas of all linear isomorphisms

$$\{(V, \phi) \mid \phi \colon V \to \mathbb{R}^n \text{ a linear isomorphism}\}.$$

Example 4.11. The charts (U_0, ψ_0) and (U_1, ψ_1) are compatible and have domains that cover S^2 so they make it into a manifold. It is not difficult to show that we cannot make S^2 into a manifold with only one chart (S^2, χ) if we require that χ is continuous. Indeed if χ is continuous then because S^2 is compact we must have $\chi(S^2) \subseteq \mathbb{R}^n$ compact and hence closed but $\chi(S^2)$ is open so this is not possible unless $\chi(S^2) = \mathbb{R}^n$ but then it is not compact.

Example 4.12. Consider the set $\mathbb{R}P_n$ of all lines through the origin in $\mathbb{R}^{n+1}$. We shall show that this is a manifold. This manifold is called *real projective space* of dimension n. If $x = (x^0, \dots, x^n)$ is a *non-zero* vector in $\mathbb{R}^{n+1}$ we denote by $[x] = [x^0, \dots, x^n]$ the line through x. In other words $[x]$ is the one-dimensional subspace spanned by x. The numbers $x = (x^0, \dots, x^n)$ are often called the *homogeneous co-ordinates* of the line $[x]$. It is important to note that they are not uniquely determined by knowing the line. Indeed we have that $[x] = [y]$ if and only if there is a non-zero real number λ such that $x = \lambda y$. Define subsets $U_i \subseteq \mathbb{R}P_n$ by

$$U_i = \{[x] \in \mathbb{R}P_n \mid x^i \neq 0\}$$

for each $i = 0, \dots, n$ and notice that these subsets cover all of $\mathbb{R}P_n$. Define maps $\psi_i \colon U_i \to \mathbb{R}^n$ by

$$\psi_i([x^0, \dots, x^n]) = \left(\frac{x^0}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^n}{x^i} \right).$$

Notice that we need to check that these maps are well-defined but that follows from the fact that $[x] = [y]$ only if x is a scalar multiple of y. It also straightforward to check that the ψ_i are bijections onto $\mathbb{R}^n$ and hence define co-ordinates. Lastly it is straightforward to check that these co-ordinate charts are all compatible and hence make $\mathbb{R}P_n$ into a manifold.

Exercise 4.2. Repeat example (4.12) for $\mathbb{C}^n$ to define $2n$ dimensional complex projective space $\mathbb{C}P_n$ as the space of complex lines through zero in $\mathbb{C}^{n+1}$.

Exercise 4.3. Use your previous calculation for the n-sphere to show how to make it a manifold with two co-ordinate charts.

We need to now deal with a technical problem raised by the definition of atlas. We often want to work with co-ordinate charts that are not in the atlas $\mathcal{A}$ used to define the differentiable structure. For example if $M = \mathbb{R}^2$ we might take $\mathcal{A} = \{1_{\mathbb{R}^n}\}$. Then in a particular problem we might want to work with polar co-ordinates. But are they somehow compatible with the differentiable structure already imposed by $\mathcal{A}$? The definition of what compatibility is in this sense is easy. We could say that another co-ordinate chart is compatible with the given atlas if when we add it to the atlas we still have an atlas. In other words it is compatible with all the charts already in the atlas. We will take a different, but equivalent, approach via the notion of a *maximal* atlas containing $\mathcal{A}$ to explain these notions. We define;

Definition 4.5 (Maximal atlas)**.** An atlas $\bar{\mathcal{A}}$ for a set M is a *maximal atlas containing* $\mathcal{A}$ if $\mathcal{A} \subseteq \bar{\mathcal{A}}$ and for any other atlas $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$ we have $\mathcal{B} \subseteq \bar{\mathcal{A}}$.

We then have

Proposition 4.6. *For any atlas $\mathcal{A}$ on a set M there is a unique maximal atlas $\bar{\mathcal{A}}$ containing $\mathcal{A}$. The maximal atlas consists of every chart compatible with all the charts in $\mathcal{A}$.*

Proof. Define the set $\bar{\mathcal{A}}$ to the set of all charts which are compatible with every chart in $\mathcal{A}$. Then clearly if $\mathcal{B}$ is another atlas for M with $\mathcal{A} \subseteq \mathcal{B}$ then we must have $\mathcal{B} \subseteq \bar{\mathcal{A}}$. What is not immediate is that $\bar{\mathcal{A}}$ is an atlas. The problem is that we do not know that the charts we have added to $\mathcal{A}$ to form $\bar{\mathcal{A}}$ are compatible with each other. So let (U, ψ) and (V, χ) be charts in $\bar{\mathcal{A}}$. We need to show that (U, ψ) is compatible with (V, χ). Recall from the definition that this is true if the sets $\psi(U \cap V)$ and $\chi(U \cap V)$ are open and

$$\chi \circ \psi^{-1}\big|_{\psi(U\cap V)} : \psi(U \cap V) \to \chi(U \cap V)$$

is a diffeomorphism. To prove this it suffices to show that for every x in $U \cap V$ we can find a W with $x \in W \subseteq U \cap V$ such that $\psi(W)$ and $\chi(W)$ are open and such that

$$\chi \circ \psi^{-1}\big|_{\psi(W)} : \psi(W) \to \chi(W)$$

is a diffeomorphism.

To find W choose a co-ordinate chart (Z, ϕ) in $\mathcal{A}$ with $x \in Z$. This is possible as the domains of the charts in an atlas cover M. Then let $W = U \cap V \cap Z$. Now (U, ψ) is compatible with (Z, ϕ) so that $\phi(U \cap Z)$ is open. Similarly $\phi(V \cap Z)$ is open so that

$$\phi(W) = \phi(U \cap Z) \cap \phi(V \cap Z)$$

is open. Using compatibility again we that

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap Z)} : \phi(U \cap Z) \to \psi(U \cap Z)$$

is a diffeomorphism and hence a homeomorphism so that

$$\psi(W) = \psi \circ \phi^{-1}(\phi(W))$$

is open as required. A similar argument shows that $\chi(W)$ is open. Then the chain rule shows that

$$\chi \circ \psi^{-1}\big|_{\psi(W)} = (\chi \circ \phi^{-1}\big|_{\phi(W)}) \circ (\phi \circ \psi^{-1}\big|_{\psi(W)})$$

is a diffeomorphism. ∎

Finally we have

Definition 4.7. If M is a manifold with atlas $\mathcal{A}$ we define a *co-ordinate chart on the manifold* $(M, \mathcal{A})$ to be a co-ordinate chart on the set M which is in the maximal atlas $\bar{\mathcal{A}}$.

It should be noted that having defined a maximal atlas we tend not to refer to it very much. We usually say (U, ψ) is a co-ordinate chart on a manifold M rather than (U, ψ) is a member of the atlas $\mathcal{A}$ for a manifold $(M, \mathcal{A})$. The situation is similar to that for a topological space X with topology $\mathcal{T}$. We rarely refer to the topology $\mathcal{T}$ by name. We say U is an open subset of X rather than U is in the topology defining the open sets of X.

Exercise 4.4. Consider the sphere S^n again. Define

$$U_i^+ = \{x \in S^n \mid x^i > 0\}$$

and define $\psi_i^+ : U_i^+ \to \mathbb{R}^n$ by

$$\psi_i^+(x^1, \ldots, x^{n+1}) = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}).$$

Show that (U_i^+, ψ_i^+) is a co-ordinate chart for S^n. Similarly define

$$U_i^- = \{x \in S^n \mid x^i < 0\}$$

and define $\psi_i^- : U_i^+ \to \mathbb{R}^n$ by

$$\psi_i^-(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1}).$$

Again show that (U_i^-, ψ_i^-) is a co-ordinate chart for S^n.

Show that

$$\{(U_i^+, \psi_i^+), (U_i^-, \psi_i^-) \mid i = 1, \dots, n+1\}$$

is an atlas for S^n.

Exercise 4.5. Show that the atlases in Exercises 4.3 and 4.4 define the same maximal atlas on S^n.

Exercise 4.6. Show that if M_1 and M_2 are manifolds then there is a natural way of making $M_1 \times M_2$ into a manifold so that $\dim(M_1 \times M_2) = \dim(M_1) + \dim(M_2)$.

4.2. **Linear manifolds.** There are many similarities between manifolds and vector spaces. Choosing co-ordinates is much like choosing a basis. Although this isn't a common approach to vector spaces it is useful as an example and for defining the linear structure on the tangent space of a manifold.

Definition 4.8. Define *linear co-ordinates* ψ on a set V to be a bijection $\psi \colon V \to \mathbb{R}^n$.

Definition 4.9. Define two sets of linear co-ordinates ψ and χ to be *linearly compatible* if $\psi \circ \chi^{-1}$ is a linear isomorphism.

It is straightforward to prove that linear compatibility is an equivalence relation. We define

Definition 4.10. A *linear atlas* on a set V is an equivalence class of linear co-ordinates.

Definition 4.11. A *linear manifold* is a set V with a choice of linear atlas.

We can define an addition and scalar multiplication on V by choosing some linear co-ordinates ψ from the linear atlas and defining

$$av + bw = \psi^{-1}(a\psi(v) + b\psi(w))$$

where a and b are real numbers and v and w are elements of V. We have to check that this is *well-defined* that is it is independent of the choice of ψ from the equivalence class. If χ is another choice then we have

$$\begin{aligned} av + bw =& \psi^{-1}(a\psi(v) + b\psi(w)) \\ =& \psi^{-1}(a\psi(\chi^{-1} \circ \chi(v)) + b\psi(\chi^{-1} \circ \chi(w))) \\ =& \psi^{-1}(a(\psi \circ \chi^{-1})(\chi(v)) + b(\psi \circ \chi^{-1})(\chi(w))) \\ =& \psi^{-1}(\psi \circ \chi^{-1})(a\chi(v) + b\chi(w)) \\ =& \chi^{-1}(a\chi(v) + b\chi(w)) \end{aligned}$$

where in moving from the third to the fourth lines we use the fact that $\psi \circ \chi^{-1}$ is linear. We have proved:

Proposition 4.12. *A linear manifold has a natural vector space structure which makes all of the linear co-ordinates linear isomorphisms.*

Because of Proposition 4.12 the theory of linear manifolds is really the theory of vector spaces. However it is an amusing exercise to translate everything in the theory of vector spaces into the linear manifold setting. For example a function $f \colon V \to \mathbb{R}$ is linear if $f \circ \psi^{-1} \colon \mathbb{R}^n \to \mathbb{R}$ is linear for some choice of linear co-ordinates ψ. It is then easy to prove that $f \circ \chi^{-1} \colon \mathbb{R}^n \to \mathbb{R}$ is linear for any choice of linear co-ordinates χ. Indeed we just note that

$$f \circ \chi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \chi^{-1}).$$

4.3. **Topology of a manifold.** Often a manifold is defined as a topological space and the domains of the charts are required to be open sets and the co-ordinates homeomorphisms. This is really superfluous as the topology is forced once we have chosen the atlas. Given a manifold M we define a subset $W \subseteq M$ to be open if for every $x \in W$ there is a chart with domain U such that $x \in U \subseteq W$. We need to show that such a definition of open sets defines a topology on M. The only problem is showing that the intersection of two open sets is open.

Lemma 4.13. *Let (U, ψ) be a co-ordinate chart on a manifold M and let $W \subseteq U$ be such that $\psi(W)$ is open. Then $(W, \psi|_W)$ is a co-ordinate chart.*

Proof. Let (V, χ) be a co-ordinate chart. By compatibility we have that $\psi(U \cap V)$ and $\chi(U \cap V)$ are open and $\chi \circ (\psi^{-1})\big|_{\psi(U\cap V)}$ is a diffeomorphism and hence a homeomorphism. It follows that $\psi(W \cap V) = \psi(W) \cap \psi(U \cap V)$ is open. Moreover

$$\chi(W \cap V) = \chi \circ (\psi^{-1})\big|_{\psi(U\cap V)} (\psi(W \cap V))$$

is open. Also the map $\chi \circ (\psi^{-1})\big|_{\psi(W\cap V)}$ is the restriction of $\chi \circ (\psi^{-1})\big|_{\psi(U\cap V)}$ so smooth. Similarly $\psi \circ (\chi^{-1})\big|_{\chi(W\cap V)}$ is the restriction of $\psi \circ (\chi^{-1})\big|_{\chi(U\cap V)}$ so smooth and hence a diffeomorphism. So $(W, \psi|_W)$ and (V, χ) are compatible. Hence $(W, \psi|_W)$ is an co-ordinate chart. □

Note that this shows that the domains of co-ordinate chart are a basis for a topology on the manifold.

Proposition 4.14. *The collection of open sets defined above forms a topology on M.*

Proof. Clearly the empty set and M are open. Let x be in an infinite union of open sets. Then x is in one of the open sets W say. So there is a co-ordinate chart (U, ψ) with $x \in U \subset W$. But then U is in the infinite union so the infinite union is open. Let $x \in W_1 \cap W_2$ where W_1 and W_2 are open. So there are co-ordinate charts (U_1, ψ_1) and (U_2, ψ_2) with $U_1 \subset W_1$ and $U_2 \subset W_2$. But by compatibility $\psi_1(U_1 \cap U_2)$ is open so by the Lemma $(U_1 \cap U_2, \psi_1|_{U_1\cap U_2})$ is a co-ordinate chart. Hence $W_1 \cap W_2$ is open. □

We call this the manifold topology. Using this we can show.

Proposition 4.15. *If M is a manifold with the manifold topology and (U, ψ) is a co-ordinate chart then $\psi \colon U \to \psi(U)$ is a homeomorphism.*

Proof. We know that $\psi \colon U \to \psi(U)$ is a bijection. Let $V \subset \psi(U)$ be open and $W = \psi^{-1}(V)$. Then from the Lemma we have that $(W, \psi|_W)$ is a chart so that W is open.

On the other hand assume that $W \subset U$ is open in the manifold topology. We want to show that $\psi(W)$ is open. Let $\psi(w) \in \psi(W)$. As W is open there must be a chart (V, χ) with $w \in V \subset W \subset U$. By compatibility $\psi(V) = \psi(V \cap U)$ is open. But $\psi(w) \in \psi(V) \subset \psi(W)$ so that $\psi(W)$ is open. □

Then we can prove also

Lemma 4.16. *Let (U, ψ) be a co-ordinate chart on a manifold M and let $W \subseteq U$ be open. Then $(W, \psi|_W)$ is a co-ordinate chart.*

Readers familiar with the notion of a basis for a topology will realise that we are claiming here that the set of all domains of coordinate charts in a maximal atlas is a basis for a topology on M. It is this topology we have described above.

We will in general require a manifold to be Hausdorff and paracompact in the topology we have defined.

Now that we have defined the topology of a manifold we can discuss its dimension. Each co-ordinate function has as range some $\mathbb{R}^d$. From the definition of compatibility it is clear that d is constant on the connected components of M. We shall go further and assume that our manifolds are such that this number d is constant on all of M. We call this number the *dimension* of the manifold.

Notice that the approach we have taken of letting the atlas define the topology allows some pathologies; for example the following.

Example 4.13. Let $M = \mathbb{R}^2$ and for each $t \in \mathbb{R}$ define a chart (U_t, ψ_t) by taking $U_t = \{(x,t) \mid x \in \mathbb{R}\}$ and define $\psi_t \colon U_t \to \mathbb{R}$ by $\psi_t(x,t) = x$. Then $\mathcal{A} = \{(U_t, \psi_t) \mid t \in \mathbb{R}\}$ is an atlas making $\mathbb{R}^2$ a one-dimensional manifold. However the topology it defines on $\mathbb{R}^2$ is not the usual one as it has horizontal lines as open sets.

If M is already a topological space we can consider the question of how many different maximal atlases there are which give rise to that topology. This is a complicated way of asking how many differentiable structures M has. If the M has a natural differentiable structure on it the other different ones are usually called *exotic*. It was a surprise to mathematicians in 1956 when Milnor showed that the seven sphere S^7 has exotic differentiable structures as do some other spheres. It was even more of a surprise in 1982 when Freedman used results of Donaldson to show that $\mathbb{R}^4$ has uncountably many different differentiable structures. If $n \neq 4$ then $\mathbb{R}^n$ has no exotic differentiable structures.

5. Smooth functions on a manifold.

Definition 5.1. Let M and N be manifolds and $f \colon M \to N$. We say that f is smooth if for all $m \in M$ there are charts (U, ψ) on M with $m \in U$ and (V, χ) on N with $f(U) \subseteq V$ such that

$$\chi \circ f|_U \circ \psi^{-1} \colon \psi(U) \to \chi(V)$$

is smooth.

Example 5.1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $f \colon U \to V$. If f is smooth in the usual sense and we take $\psi = \mathrm{id}_U$ and $\chi = \mathrm{id}_V$ then

$$\mathrm{id}_V \circ f|_U \circ \mathrm{id}_U^{-1} \colon \mathrm{id}_U(U) \to \mathrm{id}_V(V)$$

is just $f \colon U \to V$ so that f is smooth as a map between manifolds.

Of course it would be useless if the definition of smooth function depended on covering M and N with particular choices of charts and not surprisingly it doesn't. We have the following Proposition which, like so many of these results, depends on the chain rule.

Proposition 5.2. *Let $f \colon M \to N$ be a smooth map between manifolds. Let (U, ψ) be a chart on M and (V, χ) a chart on N with $f(U) \subseteq V$. Then*

$$\chi \circ f|_U \circ \psi^{-1} \colon \psi(U) \to \chi(V)$$

is smooth.

Proof. It is enough to show that for any $\psi(m) \in \psi(U)$ there is an open set $W \subset \psi(U)$ such that $(\chi \circ f|_U \circ \psi^{-1})\big|_W$ is smooth. From the definition of smooth we can find charts (U_1, ψ_1) on M with $m \in U_1$ and (V_1, χ_1) on N with $f(U_1) \subseteq V_1$ such that

$$\chi_1 \circ f|_{U_1} \circ \psi_1^{-1} \colon \psi_1(U_1) \to \chi_1(V_1)$$

is smooth. Let $W = \psi(U \cap U_1$ which is open by compatibility. Then

$$(\chi \circ f|_U \circ \psi^{-1})\big|_W = (\chi \circ \chi_1^{-1}\big|_{\chi_1(V \cap V_1)}) \circ (\chi_1 \circ f|_{U_1} \circ \psi_1^{-1}) \circ (\psi_1 \circ \psi^{-1}\big|_{\psi(U \cap U_1)}).$$

which is smooth by the chain rule. □

The chain rule also extends to the chain rule for manifolds.

Proposition 5.3. *Let $f \colon M \to N$ and $g \colon N \to K$ be smooth maps between manifolds. Then $g \circ f \colon M \to K$ is a smooth map between manifolds.*

Proof. Exercise. □

Exercise 5.1. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \exp\left(\frac{-1}{1-x^2}\right) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

and show that h is smooth. By integrating h find a smooth function $g\colon \mathbb{R} \to \mathbb{R}$ with the property that $g(x)$ is zero for $x < -1$ and $g(x)$ is one for $x > 1$. Show that for any $\epsilon > \delta > 0$ there is a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ with $\phi(x)$ equal to zero if $\|x\| > \epsilon$ and ϕ equal to one if $\|x\| < \delta$. Now consider a manifold M and a point x. By using co-ordinates show that if U is any open subset of M containing x then there are open subsets U_1 and U_2 with $x \in U_1 \subseteq U_2 \subseteq U$ and a smooth function $f\colon M \to \mathbb{R}$ with f equal to 1 on all of U_1 and equal to zero outside of U_2. Such a function f is called a *bump function*.

Exercise 5.2. Let x be a point in a manifold M. Let X_x be the set of all pairs (U, f) where U is a open set containing x and $f\colon U \to \mathbb{R}$ is a smooth function. Define a relation on X_x by saying that $(U, f) \simeq (V, g)$ if there is an open set W with $x \in W \subseteq U \cap V$ and $f|_W = g|_W$. In other words 'f and g agree in some neighbourhood of x'. Show that this an equivalence relation. Equivalence classes are called *germs* at x and the set of them we will denote by G_x. Show that G_x is an algebra under pointwise addition, scalar multiplication and multiplication. If $f \in C^\infty(M, \mathbb{R})$ the algebra of all smooth functions on M, then for every x in M we can consider the germ at x containing (M, f). Show that the map this induces $C^\infty(M, \mathbb{R}) \to G_x$ is onto. [Hint: Use 5.1.]

Definition 5.4. A function $f\colon M \to N$ between manifolds is a *diffeomorphism* if it is smooth and invertible and the inverse is smooth.

Exercise 5.3. Define a map $F\colon S^2 \to \mathbb{C}P_1$ by

$$F(x, y, z) = [x + iy, 1 - z].$$

By using the co-ordinates show that this map is well defined as $z \to 1$ and that it is, in fact, a diffeomorphism.

6. The tangent space.

We will also be interested in smooth functions of a single real variable into a manifold or paths. Later we will give a more general definition but it suffices for now to assume that the image of the path lies in the domain of a single co-ordinate chart. We have

Definition 6.1. If x is a point of a manifold and $\gamma\colon (-\epsilon, \epsilon) \to M$ is smooth we say that γ is a *path* through x if $\gamma(0) = x$.

Example 6.1. If x is a point in $\mathbb{R}^n$ and v is a vector in $\mathbb{R}^n$ then the function

$$t \mapsto x + tv$$

is a path through x.

Example 6.2. If $x \in S^2$ and $v \in \mathbb{R}^3$ with $\langle x, v\rangle = 0$ then

$$t \mapsto \frac{x + tv}{\|x + tv\|}$$

is a path in S^2 through x.

Most of the theory of calculus on manifolds needs the idea of tangent vectors and tangent spaces. The name 'tangent vector' comes of course from examples like $S^2 \subseteq \mathbb{R}^3$ where a tangent vector at $x \in S^2$ is a vector in $\mathbb{R}^3$ tangent to the sphere which in that particular case means orthogonal to x. However in the case of a general manifold M it does not come to us sitting inside some $\mathbb{R}^N$ and we have to work a little harder to develop a notion of tangent vector.

Although we do not have a notion of tangent vector yet we do have the notion of a smooth path in a manifold. Let us see what this does for us in $\mathbb{R}^n$. For simplicity we will assume that the domain of the path is all of $\mathbb{R}$. In that case if $f\colon \mathbb{R}^n \to \mathbb{R}$ and a path $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ with $\gamma(0) = x$ then we can consider

$$(f \circ \gamma)'(0)$$

the rate of change of f along γ as we go through 0. By the chain rule we can write this as

$$(f \circ \gamma)'(0) = df(x)(\gamma'(0))$$

where $\gamma'(0)$ is the tangent vector to γ at $t = 0$. Notice that this equation tells us the $(f \circ \gamma)'(0)$ depends on γ only through $\gamma'(0)$, that is if we replace γ by another path ρ with $\rho(0) = x$ and $\rho'(0) = \gamma'(0)$ then

$$(f \circ \gamma)'(0) = (f \circ \rho)'(0).$$

If $\gamma\colon (-\epsilon, \epsilon) \to M$ is a path through m and (U, ψ) is a chart containing m then $\gamma^{-1}(U)$ is an open set containing 0. So we can find ϵ' so that $\gamma(-\epsilon', \epsilon') \subset U$ and hence $\gamma|_{(-\epsilon',\epsilon')}$ can be composed with ψ. Consider

$$(\psi \circ \gamma|_{(-\epsilon',\epsilon')})'(0) \in \mathbb{R}^n.$$

From the definition of the derivative as a limit this is clearly independent of the choice of ϵ' so for simplicity we denote it by $(\psi \circ \gamma)'(0)$.

Notice that we have

Lemma 6.2. *If (U, ψ) and (V, χ) are charts with $x \in V$ then*

$$(\chi \circ \gamma)'(0) = d(\chi \circ \psi^{-1}|_{\psi(U \cap V)})(\chi(x))((\psi \circ \gamma)'(0)).$$

We these results we can define the important notion of tangency of paths.

Definition 6.3 (Tangency)**.** Let $\gamma\colon (-\epsilon, \epsilon) \to M$ and $\rho\colon (-\delta, \delta) \to M$ be paths though a point x. We say that γ and ρ are *tangent* (at 0) if there is a co-ordinate chart (U, ψ) with

$$(\psi \circ \gamma)'(0) = (\psi \circ \rho)'(0).$$

Again we have the usual lemma.

Lemma 6.4. *Let $\gamma\colon (-\epsilon, \epsilon) \to M$ and $\rho\colon (-\delta, \delta) \to M$ be paths though a point x which are tangent. If (V, χ) is a chart with $x \in V$ then*

$$(\chi \circ \gamma)'(0) = (\chi \circ \rho)'(0).$$

Proof. Follows from Lemma 6.2. ∎

We have a function that sends any path γ through x to a vector $(\psi \circ \gamma)'(0)$ so it follows from Exercise 2.5 that tangency is an equivalence relation. The equivalence classes are called tangent vectors (although we have not yet shown that they are vectors). The equivalence class containing a path γ is denoted by $\gamma'(0)$ or $t_0(\gamma)$. If X is a tangent vector and $\gamma \in X$ then we usually say that X is tangent to γ rather than that γ is an element of X. The set of all tangent vectors at x we denote by T_xM. We want to show now that T_xM has the structure of a vector space.

If (U, ψ) is a chart containing $x \in M$ then we have defined a map

$$d\psi(x)\colon T_xM \to \mathbb{R}^n$$

by

$$d\psi(x)(\gamma'(0)) = (\psi \circ \gamma)'(0).$$

By definition of tangency this map is injective we want to prove

Proposition 6.5. *The map $d\psi(x)$ is a bijection.*

Proof. As we have already noted it suffices to show that this map is onto. Let (U, ψ) be a chart about x. If v is a vector in $\mathbb{R}^n$ then $t \mapsto \psi(x) + tv$ is a path in $\mathbb{R}^n$ with tangent vector v. Because $\psi(U)$ is open we can find an $\epsilon > 0$ such that if $|t| < \epsilon$ then $\psi(x) + tv \in \psi(U)$. Then we can define $\gamma\colon (-\epsilon, \epsilon) \to M$ by

$$\gamma(t) = \psi^{-1}(\psi(x) + tv).$$

Then we have $\psi \circ \gamma(t) = \psi(x) + tv$ so that $(\psi \circ \gamma)'(0) = v$. ∎

Lemma 6.6. *If χ and ψ are co-ordinates on M and γ is a path through x then*

$$d\chi(x) = d(\chi \circ \psi^{-1})(\psi(x)) \circ d\psi(x).$$

Proof. Follows from Lemma 6.2. ∎

From the discussion in the previous section the maps $d\chi(x)$ define linear co-ordinates on T_xM and hence by Proposition 4.12 T_xM has a unique vector space structure which makes all the maps $d\psi(x)$ linear isomorphisms. To spell this out in detail we note the following. If $X, Y \in T_xM$ and $\alpha, \beta \in \mathbb{R}$ then

$$\alpha X + \beta Y = (d\psi(x))^{-1}(\alpha\ d\psi(x)(X) + \beta\ d\psi(x)(Y))$$

for any choice of co-ordinates (U, ψ) containing x.

Example 6.3. As always the first example is $M = \mathbb{R}^n$. In that case we have a preferred set of co-ordinates. These are just the identity. So two paths γ and ρ are tangent if and only if $\gamma'(0) = \rho'(0)$. In other words two paths are tangent if they have the same tangent vector at x. Notice also that if v is any vector there is a preferred path whose tangent vector is v. That is the straight line $t \mapsto x + tv$. So in the case of $\mathbb{R}^n$ there is no reason to introduce all the extra machinery of equivalence classes of paths.

Example 6.4. The second example is $M = V$ a finite dimensional vector space. Notice that if γ is a path through v in V then we can make sense of of the derivative of γ at 0 directly by

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}.$$

Of course $\gamma'(0)$ defined in this way is a vector in V whereas above we have defined $\gamma'(0)$ as an equivalence class of paths. This correspondence defines an isomorphism

$$\begin{aligned} T_v(V) &\to V \\ t_0(\gamma) &\mapsto \gamma'(0) \end{aligned}$$

Notice that the inverse to this map sends a vector w to the tangency class of the straight line $w \mapsto v + tw$ and that each tangency class $t_0(\gamma)$ contains a unique straight line $t \mapsto \gamma(0) + t\gamma'(0)$. Again in this case the extra machinery of equivalence classes of paths adds nothing to what we already know.

In the introduction to this section we remarked on the case of the two-sphere which is a submanifold of $\mathbb{R}^3$. We will return to submanifolds shortly but first we need to consider the notion of the derivative of a function.

6.1. **The derivative of a function.** Recall that a function $f : M \to \mathbb{R}$ is smooth if we can cover M with co-ordinates (U, ψ) such that $f \circ \psi^{-1} : \psi(U) \to \mathbb{R}$ is smooth . If γ is a smooth path through $x \in M$ then if follows from the chain rule that

$$f \circ \gamma = (f \circ \psi^{-1}) \circ (\psi \circ \gamma)$$

is smooth. Hence we can differentiate the function $f \circ \gamma$ at $t = 0$. By the chain rule we have that

$$(f \circ \gamma)'(0) = d(f \circ \psi^{-1})(\psi(x))((\psi \circ \gamma)'(0)).$$

It follows that $(f \circ \gamma)'(0) = (f \circ \rho)'(0)$ if ρ and γ are in the same tangency class. Hence if $X = t_0(\gamma)$ is a tangent vector in T_xM we can define

$$df(x)(X) = (f \circ \gamma)'(0).$$

We call this the *rate of change* of f in the direction X. Notice that we can calculate $df(x)(X)$ without explicit reference to the path γ by the formula

$$df(x)(X) = d(f \circ \psi^{-1})(\psi(x))d\psi(x)(X).$$

As we vary the tangent vector X we define a map

$$df(x) : T_xM \to \mathbb{R}$$

called the *differential* of f at x. This map satisfies the formula

$$df(x) = d(f \circ \psi^{-1})(\psi(x)) \circ d\psi(x)$$

and hence, being a composition of linear maps, is linear.

We call the set of linear maps from T_xM to $\mathbb{R}$ the *cotangent* space to M at x and denote it by T_x^*M. So we have

$$df(x) \in T_x^*M.$$

Elements of T_x^*M are also called *one-forms*.

If $f\colon M \to V$ is a smooth function into a vector space we can also define $df(x)$ by

$$df(x) = (f \circ \gamma)'(0)$$

which is a vector in V. The one-form $df(x)$ is then an example of a *vector space valued one-form.*

6.2. **Co-ordinate tangent vectors and one-forms.** Let (U, ψ) be a set of co-ordinates on M where $\psi = (\psi^1, \ldots, \psi^n)$. Then each of the component functions ψ^i is a real function so we can define n one-forms $d\psi^i(x) \in T_x^*M$ called the *co-ordinate one-forms.*

We have seen that

$$d\psi(x)\colon T_xM \to \mathbb{R}^n$$

is a linear isomorphism. We denote by

$$\begin{aligned}\frac{\partial}{\partial \psi^i}(x) &= d\psi(x)^{-1}(e_i)\\ &= t_0(s \mapsto \psi^{-1}(\psi(x) + se_i))\end{aligned}$$

the image under the inverse of this map of the standard basis vector e_i in $\mathbb{R}^n$. We call the set of these the *basis of co-ordinate tangent vectors.* Consider what happens when we apply $d\psi^i(x)$ to $\partial/\partial\psi^j(x)$. We have

$$\begin{aligned}d\psi^i(x)\left(\frac{\partial}{\partial \psi^j}(x)\right) &= \frac{d}{dt}(\,\psi^i(\psi^{-1}(\psi(x) + se_j))\big|_{s=0}\\ &= \frac{d}{dt}(\,\psi^i(x) + s\delta^i_j)\big|_{s=0}\\ &= \delta^i_j.\end{aligned}$$

It follows from linear algebra that $d\psi^1(x), \ldots, d\psi^n(x)$ is a basis of T_xM and, in fact, the *dual basis* to the basis $\partial/\partial\psi^1(x), \ldots, \partial/\partial\psi^n(x)$.

6.3. **How to calculate.** It is useful for calculations to know how to expand various quantities in these co-ordinate bases. First let f be a smooth function on M then we must have

$$df(x) = \sum_{i=1}^n a_i d\psi^i(x)$$

for some real numbers a_i. This is just linear algebra as is the fact that if we apply both sides of this equation to $\partial/\partial\psi^j(x)$ and use the dual basis relation we deduce that

$$a_i = df(x)\left(\frac{\partial}{\partial \psi^i}(x)\right)$$

we *define*

$$\frac{\partial f}{\partial \psi^i}(x) = df(x)\left(\frac{\partial}{\partial \psi^i}(x)\right)$$

and hence have the formula:

$$df(x) = \sum_{i=1}^n \frac{\partial f}{\partial \psi^i}(x) d\psi^i(x).$$

Note that the motivation for this definition is that

$$\begin{aligned}\frac{\partial f}{\partial \psi^i}(x) &= df(x)\left(\frac{\partial}{\partial \psi^i}(x)\right)\\ &= \frac{\partial}{\partial t}[t \mapsto f(\psi^{-1}(\psi(x) + te^i)]\\ &= \frac{\partial}{\partial t}[t \mapsto f \circ \psi^{-1}(\psi(x) + te^i)]\\ &= \partial_i f \circ \psi^{-1}(\psi(x)).\end{aligned}$$

If γ is a path through x then its tangent at 0, $\gamma'(0)$ can be expanded as

$$\gamma'(0) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial \psi^i}(x).$$

Applying $d\psi^j(x)$ to both sides we deduce that

$$\begin{aligned}\gamma'(0) &= \sum_{i=1}^{n} d\psi^i(x)(\gamma'(0)) \frac{\partial}{\partial \psi^i}(x) \\ &= \sum_{i=1}^{n} (\psi^i \circ \gamma)'(0) \frac{\partial}{\partial \psi^i}(x).\end{aligned}$$

6.4. **Submanifolds.** Historically the theory of differential geometry arose from the study of surfaces in $\mathbb{R}^3$. We want to consider the more general case of submanifolds in $\mathbb{R}^n$. Recall that we have

Definition 6.7. A set $Z \subseteq \mathbb{R}^n$ is a *submanifold of dimension* d if there is a smooth map $f \colon \mathbb{R}^n \to \mathbb{R}^{n-d}$ for some d such that $Z = f^{-1}(0)$ and $df(z)$ is onto for all $z \in Z$.

We will show that any submanifold of $\mathbb{R}^n$ is a manifold by constructing co-ordinate charts. The idea is simple. At any point $z \in Z$ we define the subspace tangent to Z. This is the kernel K_z of the map $df(z)$. Then we consider orthogonal projection of K_z onto Z. The inverse of this map defines co-ordinates on an open neighbourhood of z. In fact these are a very special kind of co-ordinate. To prove this we first prove

Proposition 6.8. *Let Z be a submanifold of $\mathbb{R}^n$ of dimension d. Then at any $z \in Z$ we can find a co-ordinate chart (U, ψ) on $\mathbb{R}^n$ such that if $\psi = (\psi^1, \dots, \psi^n)$ then*

$$U \cap Z = \{x \in U \mid \psi^{d+1}(x) = \cdots = \psi^n(x) = 0\}.$$

Proof. Let K_z be the kernel of $df(z)$ and let $\pi \colon \mathbb{R}^n \to K_z$ be the orthogonal projection onto K_z. Choose a basis $v^1, \dots, v^d$ of K_z and write π_z with respect to this basis as

$$\pi(x) = \sum_{i=1}^{d} \pi^i(x) v^i.$$

Define a map $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi(x) = (\pi^1(x - z), \dots, \pi^d(x - z), f^1(x), \dots, f^{n-d}(x)).$$

Because π is a linear map it is its own derivative so we have

$$d\psi(z) = (\pi^1(z), \dots, \pi^d(z), df^1(z), \dots, df^{n-d}(z)).$$

Consider v in the kernel of $d\psi(z)$. Then $d\pi(z)(v) = 0$ and $df(z)(v) = 0$. Hence v is both orthogonal to K_z and in K_z so it must be zero. So $d\psi(z)$ is injective and hence by a dimension count surjective so a bijection. Now we can apply the inverse function theorem so there is an open set U in $\mathbb{R}^n$ such that $\psi(U)$ is open and

$$\psi_{|U} \colon U \to \psi(U)$$

is a diffeomorphism. But this just means that $(U, \psi_{|U})$ is a co-ordinate chart on $\mathbb{R}^n$. Notice that $\psi^{d+1}(x) = \cdots = \psi^n(x) = 0$ if and only if $f(x) = 0$ if and only if x is in $U \cap Z$. $\square$

Let (U, ψ) be a set of co-ordinates on $\mathbb{R}^n$ such that

$$U \cap Z = \{x \in U \mid \psi^{d+1}(x) = \cdots = \psi^n(x) = 0\}.$$

Then consider $(U \cap Z, \bar{\psi})$ where $\bar{\psi} = (\psi^1_{|U\cap Z}, \dots, \psi^d_{|U\cap Z})$. This is a co-ordinate chart. The only thing to check is that because $\psi(U)$ is open then

$$\bar{\psi}(U \cap Z) = \{x \in \mathbb{R}^d \mid (x^1, \dots, x^d, 0, \dots, 0) \in \psi(U)\}$$

is open. This is an elementary fact about the topology of $\mathbb{R}^n$. Consider two such co-ordinate charts (U, ψ) and (V, χ). We will prove that $(U \cap Z, \bar{\psi})$ and $(V \cap Z, \bar{\chi})$ are compatible. This follows essentially from the fact that (U, ψ) and (V, χ) are compatible. First we note that

$$\bar{\chi}(U \cap V \cap Z) = \{x \in \mathbb{R}^d \mid (x^1, \dots, x^d, 0, \dots, 0) \in \chi(U \cap V)\}$$

and, again, this is open. For the smoothness of the co-ordinate change map we note that

$$\bar{\psi}^i \circ \bar{\chi}^{-1}_{|\bar{\chi}(U \cap V \cap Z)}(x) = \psi^i \circ \chi^{-1}_{|\chi(U \cap V)}(x, 0)$$

where $(x, 0) = (x^1, \dots, x^d, 0, \dots, 0)$. Hence the result follows.

We have now proved

Theorem 6.9. *If Z is a submanifold the set of charts above is an atlas.*

This results gives us a lot of examples of submanifolds:

Example 6.5 (Spheres). Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$ defined as the set of points x whose length $\|x\|$ is equal to one. Here

$$\|x\|^2 = \sum_{i=1}^{n} (x^i)^2.$$

We can prove it is a submanifold of $\mathbb{R}^n$ and hence a manifold by considering the map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x) = \|x\|^2 - 1.$$

Clearly this is smooth and has zero set Z equal to the sphere S^n. To check that the derivative is onto note that

$$df(x)(v) = 2\langle x, v\rangle$$

and is a linear map onto a one-dimensional space so to show it is onto we just need to show that it is not equal to the zero linear map.

Example 6.6 (The orthogonal group). The orthogonal group is the group of all linear transformations of $\mathbb{R}^n$ that preserve the usual inner product on $\mathbb{R}^n$. We shall think of it as a group of n by n matrices:

$$O(n) = \{X \mid XX^t = I\}.$$

We can identify the set of all n by n matrices with $\mathbb{R}^{n^2}$. There are various ways of doing this. So as to be concrete let us assume we have done it by writing down the rows one after the other. With this identification in mind define a smooth map $f\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ by $f(X) = XX^t - I$. It is clear we have $f^{-1}(0) = O(n)$. Define the linear subspace $S \subseteq \mathbb{R}^{n^2}$ to be the set of all symmetric matrices. This can be identified with $\mathbb{R}^d$ where $d = n(n+1)/2$. It is easy to check that f takes its values in S so we will think of f as a smooth map $f\colon \mathbb{R}^{n^2} \to \mathbb{R}^d = S$.

We want to calculate $df(X)$ the derivative of f at a matrix X. By differentiating the path $t \mapsto X + tY$ we obtain

$$df(X)(Y) = YX^t + XY^t.$$

If B is any symmetric matrix then it is easy to check that $df(X)((1/2)BX) = B$ if we use that fact that $XX^t = 1$ and $B = B^t$. We have therefore shown that $df(X)$ is onto for any $X \in O(n)$ so that $O(n)$ is a submanifold of dimension $n^2 - d = n(n-1)/2$.

Exercise 6.1. Show that the set defined by the equation

$$r^2 - z^2 = (\sqrt{x^2 + y^2} - R)^2$$

is a smooth submanifold of $\mathbb{R}^3$ if R and r are real numbers with $0 < r < R$.

Exercise 6.2. Show that the following subset of $\mathbb{R}^3$ is a submanifold:

$$Q = \{(x, y, z) \mid x^2 + y^2 + z^2 = 9 \quad \text{and} \quad x + y + z = 3\}.$$

6.5. **Tangent space to a submanifold of $\mathbb{R}^n$.** If Z is a submanifold of $\mathbb{R}^n$ then there is a natural notion of the plane tangent to Z at any point x independent of abstract notions such as equivalence classes of paths and co-ordinates. It is just the subspace of $\mathbb{R}^n$ tangential to Z at x. More precisely if $Z = f^{-1}(0)$ it is the kernel of $df(x)$ which we denoted by K_x. To relate this to the abstract notion of tangent vector consider a smooth path

$$\gamma\colon (-\epsilon, \epsilon) \to Z.$$

Because $Z \subseteq \mathbb{R}^n$ this is naturally a path in $\mathbb{R}^n$. We check first that this is smooth. To do this choose co-ordinates (U, ψ) for $\mathbb{R}^n$ about x satisfying

$$U \cap Z = \{x \in Z \mid \psi^{d+1}(x) = \cdots = \psi^n(x) = 0\}.$$

and denote by $\bar{\psi}$ the corresponding co-ordinates on $U \cap Z$. Smoothness of γ means that the functions $\bar{\psi}^i \circ \gamma = \psi^i \circ \gamma$ are smooth for each $i = 1, \ldots, d$. Because γ has image inside Z we also have that $\psi^i \circ \gamma = 0$ for each $i = d+1, \ldots, n$ and hence these are also smooth. So γ is a smooth path in $\mathbb{R}^n$. Consider the vector $\gamma'(0)$ in $\mathbb{R}^n$. We have that $f \circ \gamma(t) = 0$ for all t so by the chain rule $df(x)(\gamma'(0)) = 0$ so that $\gamma'(0) \in K_x$.

We can now define a map $T_x Z \to K_x$. If $X \in T_x Z$ then we choose a path γ in Z whose tangent vector at 0 is X and map X to $\gamma'(0) \in K_x$. We have to check first that this is well-defined. Let ρ be another such path and consider the co-ordinates $\bar{\psi}$. By definition we have

$$(\bar{\psi}^i \circ \gamma)'(0) = (\bar{\psi}^i \circ \rho)'(0)$$

for every $i = 1, \ldots, d$. Hence we also have

$$(\psi^i \circ \gamma)'(0) = (\psi^i \circ \rho)'(0)$$

for every $i = 1, \ldots, d$. But

$$(\psi^i \circ \gamma)'(0) = (\psi^i \circ \rho)'(0) = 0$$

for $i = d+1, \ldots, n$ so we have

$$(\psi^i \circ \gamma)'(0) = (\psi^i \circ \rho)'(0)$$

for $i = 1, \ldots, n$. Hence X maps to the same element of K_x whether we use γ or ρ. To show that this map is injective we use a similar argument. It is easy to see that this map is linear. Hence, counting, dimensions we see that this is a linear isomorphism.

We conclude that if $Z \subseteq \mathbb{R}^n$ is a submanifold and we consider the tangents to all the paths through $x \in Z$, thought of as maps into $\mathbb{R}^n$ then they span the space K_x.

6.6. **The tangent to a smooth map.** If $f\colon M \to N$ is a smooth map and $\gamma\colon (-\epsilon, \epsilon) \to M$ is a smooth path through x then $f \circ \gamma$ is a smooth path in N through $f(x)$. Moreover if we consider another path ρ which is tangent to γ then $f \circ \gamma$ and $f \circ \rho$ are tangent. To see this choose co-ordinates (U, ψ) and (V, χ) with $f(U) \subseteq V$. Assume without loss of generality that $\gamma(-\epsilon, \epsilon)$ and $\rho(-\epsilon, \epsilon)$ are in U. Then we have

$$(\chi \circ f \circ \gamma)'(0) = d(\chi \circ f \circ \psi^{-1})(\psi(x))(\psi \circ \gamma)'(0)$$

and

$$(\chi \circ f \circ \rho)'(0) = d(\chi \circ f \circ \psi^{-1})(\psi(x))(\psi \circ \rho)'(0)$$

so that $(\psi \circ \rho)'(0) = (\psi \circ \gamma)'(0)$ implies that $(\chi \circ f \circ \rho)'(0) = (\chi \circ f \circ \gamma)'(0)$ and hence $f \circ \gamma$ and $f \circ \rho$ are tangent. So associated with f there is a well-defined map from $T_x M$ to $T_{f(x)} N$ that sends $\gamma'(0)$ to $(f \circ \gamma)'(0)$. This map is denoted $T_x f$ and called the *tangent to f at x*. So we have that

$$T_x(f)(\gamma'(0)) = (f \circ \gamma)'(0).$$

Notice that the tangent map satisfies

$$T_x(f) = d\chi(f(x))^{-1} \circ d(\chi \circ f \circ \psi^{-1})(\psi(x)) \circ d\psi(x).$$

so that, being a composition of three linear maps it is, itself linear. Moreover this formula also shows that with respect to the bases of $T_x M$ and $T_{f(x)} N$ given by the co-ordinate vector fields we have

$$T_x(f)\left(\frac{\partial}{\partial \psi^i}(x)\right) = \sum_{j=1}^{n} \frac{\partial \chi^j \circ f}{\partial \psi^i}(x) \frac{\partial}{\partial \chi^j}(f(x)).$$

In other words it is given by the action of a matrix whose entries are the partial derivatives of the co-ordinate expression for f.

Example 6.7. The tangent space to $\mathbb{R}^n$ at $\psi(x)$ is just $\mathbb{R}^n$ again. The map $d\psi(x)\colon T_xM \to \mathbb{R}^n$ is just the map $T_x\psi\colon T_xM \to T_{\psi(x)}\mathbb{R}^n$.

Example 6.8. If $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map then, after identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ and $T_{F(x)}\mathbb{R}^m$ with $\mathbb{R}^m$ we see that the tangent map $T_x(F)$ is just the matrix of partial derivatives $dF(x)$.

The chain rule for smooth functions in $\mathbb{R}^n$ generalises to manifolds as follows.

Proposition 6.10 (Chain Rule)**.** *Let $f\colon M \to N$ and $g\colon N \to P$ be smooth functions. Then the map $g \circ f\colon M \to P$ is smooth and $T_x(g \circ f) = T_{f(x)}(g) \circ T_x(f)$.*

If $f\colon M \to V$ is a smooth map from a manifold to a vector space then we can define, as for a real valued function, a derivative by

$$df(x)(t_0(\gamma)) = (f \circ \gamma)'(0)$$

which is a vector on V. On the other hand we have just defined

$$T_x(f)\colon T_x(M) \to T_{f(x)}(V).$$

To understand the relation between these two notions of derivative recall from Example 6.4 that we have seen that the tangent spaces to a vector space are naturally identified with the vector space itself by differentiating each path. If we compose $T_x(f)$ with this identification $T_{f(x)}(V) \simeq V$ then we obtain $df(x)$.

6.7. **Submanifolds again.** If M is a manifold then we can define a submanifold of M by using the principal property of submanifolds in $\mathbb{R}^n$.

Definition 6.11 (Submanifolds.)**.** We say that a subset $N \subseteq M$ is a submanifold of dimension d of a manifold M of dimension m if for every $x \in N$ we can find a co-ordinate chart (U, ψ) for M with $x \in U$ and such that

$$U \cap N = \{y \in N \mid \psi^{d+1}(y) = \cdots = \psi^n(y) = 0\}.$$

Just as before we can define co-ordinates $(U \cap N, \bar{\psi})$ on N by letting

$$\bar{\psi}^i(y) = \psi^i(y)$$

for each $i = 1, \ldots, d$. Similarly we have

Proposition 6.12. *The set consisting of all the charts $(U \cap N, \bar{\psi})$ constructed in this manner is an atlas. Moreover it makes N a manifold in such a way that the inclusion map $\iota_N\colon N \to M$ defined by $\iota_N(n) = n$ is smooth.*

Because the condition for being a submanifold is local we can use the inverse function theorem as in Proposition 6.8 to prove

Proposition 6.13. *Let $f\colon M \to N$ be a smooth map between manifolds of dimension m and n respectively. Let $n \in N$ and $Z = f^{-1}(n)$. Then if T_zf is onto for all $z \in Z$ the set Z is submanifold of M. Moreover the image of $T_z(\iota_Z)$ in T_zM is precisely the kernel of T_zf.*

6.8. **Vector fields.** We have seen how to define tangent vectors at a point of a manifold. In many problems we are interested in vector fields on X, that is a choice of vector $X(x) \in T_xM$ at every point of a manifold. We need to make sense of the notion of such a vector $X(x)$ depending smoothly on x. We do this as follows. Choose a chart (U, ψ). Then at every point $x \in U$ we have a basis

$$\frac{\partial}{\partial \psi^1}(x), \ldots, \frac{\partial}{\partial \psi^n}(x)$$

of T_xM and we can expand $X(x)$ as a linear combination of these tangent vectors:

$$X(x) = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial \psi^i}(x).$$

We call the functions $X^i\colon U \to \mathbb{R}$ the *components* of the vector field with respect to the co-ordinate chart. We define

Definition 6.14. A vector field X on a manifold M is called *smooth* if its components with respect to a collection of co-ordinate charts whose domains cover M are all smooth.

We have the usual lemma.

Lemma 6.15. *If X is a smooth vector field then its components with respect to any co-ordinate chart are smooth.*

Proof. Let (U,ψ) be a co-ordinate chart and let $x \in U$. Choose a co-ordinate chart (V,χ) with $x \in V$ and such that the components of X are smooth with respect to (V,χ). Then for $x \in U$ write

$$X = \sum_{i=1}^{n} \alpha^i(x) \frac{\partial}{\partial \psi^i}(x).$$

Similarly for $x \in V$ write

$$X = \sum_{j=1}^{n} \beta^j(x) \frac{\partial}{\partial \chi^j}(x)$$

and note that by assumption $\beta^j \colon V \to \mathbb{R}$ is smooth for all $j = 1, \ldots, n$. From the results of section 6.3 we have

$$d\chi^j = \sum_{i=1}^{n} \frac{\partial \chi^j}{\partial \psi^i} d\psi^i$$

so using the property of dual bases we have

$$\frac{\partial}{\partial \psi^i} = \sum_{a=1}^{n} \frac{\partial \chi^a}{\partial \psi^i} \frac{\partial}{\partial \chi^a}.$$

Hence

$$\alpha^i(y) = \sum_{j=1}^{n} \frac{\partial \psi^i}{\partial \chi^j}(y) \beta^j(y)$$

for all $y \in U \cap V$ so that the α^i are smooth on $U \cap V$ for all $i = 1, \ldots, n$ and hence smooth on all of U. ∎

Definition 6.16 (Tangent bundle)**.** If M is a manifold we define the *tangent bundle* of M to be the union of all the tangent spaces

$$TM = \bigcup_{x \in M} T_x M$$

equipped with the map $\pi \colon TM \to M$ which maps $X \in T_x M$ to x.

Definition 6.17. If $f \colon A \to B$ is an onto function we call $s \colon B \to A$ a *section of f* or a *section of $f \colon A \to B$* if it satisfies $f \circ s = \mathrm{id}_B$.

Notice that a vector field on M is just a section of the tangent bundle $TM \to M$. If TM was a manifold then we would automatically have a definition of a smooth vector field and could avoid the need for Definition 6.14. We can achieve this as follows:

Exercise 6.3. If (U,ψ) is a chart on M let $X \in T_x M$ where $x \in U$. In other words $X \in \pi^{-1}(U) \subseteq TM$. Consider $(\psi, d\psi) \colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ defined by

$$(\psi, d\psi)(X) = (\psi(x), d\psi(x)(X)).$$

What is the image of $(\psi, d\psi)$? Show that $(\pi^{-1}(U), (\psi, d\psi))$ is a chart for TM and that the collection of all of them forms an atlas for TM making it a manifold. Show that with this differentiable structure $TM \to M$ is smooth and a section $X \colon M \to TM$ is smooth if and only if it is a smooth vector field as defined in Definition 6.14.

Classical texts on differential geometry, in particular those on tensor calculus, downplay the co-ordinates and charts and concentrate on the components of vector fields and similar tensors. Assume that M is covered by the domains of co-ordinate charts (U_α, ψ_α). For each chart (U_α, ψ_α) we write

$$X|_{U_\alpha} = \sum_{i=1}^{n} X^i_\alpha \frac{\partial}{\partial \psi^i} \tag{6.1}$$

and then as in the proof above we have that for x in the intersection of U_α and U_β we have

$$X^i_\alpha(x) = \sum_{j=1}^n \frac{\partial \psi^i_\alpha}{\partial \psi^j_\beta}(x) X^j_\beta(x). \tag{6.2}$$

The converse is also true. If we have a collection of maps $X^i_\alpha \colon U_\alpha \to \mathbb{R}$ satisfying 6.2 then we can define a vector field using 6.1 and check that it is well-defined. Classical and physics texts generally suppress the α index and also the sum by applying the Einstein summation convention. This convention is that any index that occurs in an expression in both a raised and lowered position is summed over. So a typical way of writing 6.2 would be to say that we have co-ordinates x^i and co-ordinates $x^{i'}$ and that the vector field transforms as

$$X^i = \frac{\partial x^i}{\partial x^{j'}} X^{j'}.$$

Notice that even if we do not exploit the Einstein summation convention it is a useful guide to memorising expressions like this. To apply it correctly we need to remember that the index on a co-ordinate is a superscript.

6.9. **The Lie bracket.** One use of this discussion is the definition of the Lie bracket of two vector fields. Let X and Y be two vector fields and write them locally as

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial \psi^i}$$

and

$$Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial \psi^i}.$$

so that

Then define

$$[X, Y] = \sum_{i,j=1}^n \left(X^j \frac{\partial Y^i}{\partial \psi^j} - Y^j \frac{\partial X^i}{\partial \psi^j} \right) \frac{\partial}{\partial \psi^i}.$$

We leave it as an exercise to show that this transforms as a vector field. We call $[X, Y]$ the *Lie bracket* of the vector fields X and Y. Lie is named after Sophus Lie and pronounced "lee".

7. Differential forms.

In vector calculus in $\mathbb{R}^3$ extensive use is made of the idea of vector fields and the differential operators grad, div and curl. Differential forms and their associated exterior derivative are the generalisations to higher dimensions, and manifolds of these ideas.

7.1. **The exterior algebra of a vector space.** If V is a vector space we define a *k-linear map* to be a map

$$\omega \colon V \times \cdots \times V \to \mathbb{R},$$

where there are k copies of V, which is linear in each factor. That is

$$\omega(v_1, \dots, v_{i-1}, \alpha v + \beta w, v_{i+1}, v_k) = \alpha\, \omega(v_1, \dots, v_{i-1}, v, v_{i+1}, v_k) + \beta\, \omega(v_1, \dots, v_{i-1}, w, v_{i+1}, v_k).$$

for every $i = 1, \dots, n$. We define a k-linear map ω to be *totally antisymmetric* if

$$\omega(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$$

for all vectors $v_1, \dots, v_k$ and all i. Note that it follows that

$$\omega(v_1, \dots, v, v, \dots, v_k) = 0$$

and if $\pi \in S_k$ is a permutation of k letters then

$$\omega(v_1, v_2, \dots, v_k) = \operatorname{sgn}(\pi) \omega(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation π. We denote the vector space of all k-linear, totally antisymmetric maps by $\Lambda^k(V^*)$. and call them k forms. If $k = 1$ the $\Lambda^1(V^*)$ is just V^* the space of all linear functions on V and if $k = 0$ we make the convention that $\Lambda^0(V^*) = \mathbb{R}$. We need to collect some results on the linear algebra of these spaces.

Assume that V has dimension n and that $v_1, \dots, v_n$ is a basis of V. Let ω be a k form. Then if $w_1, \dots, w_k$ are arbitrary vectors and we expand them in the basis as

$$w_i = \sum_{j=1}^{n} w_{ij} v_j.$$

then we have

$$\omega(w_1, \dots, w_k) = \sum_{j_1, \dots, j_k = 1}^{n} w_{1j_1} w_{2j_2} \dots w_{kj_k} \omega(v_{j_1}, \dots, v_{j_k})$$

so that it follows that ω is completely determined by its values on basis vectors. In particular if $k > n$ then $\Lambda^k(V^*) = 0$.

If α^1 and α^2 are two linear maps in V^* then we define an element $\alpha^1 \wedge \alpha^2$, called the *wedge product* of α^1 and α^2, in $\Lambda^2(V^*)$ by

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1).$$

More generally if $\omega \in \Lambda^p(V^*)$ and $\rho \in \Lambda^q(V^*)$ we define $\omega \wedge \rho \in \Lambda^{p+q}(V^*)$ by

$$(\omega \wedge \rho)(w_1, \dots, w_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \mathrm{sgn}(\pi)\omega(w_{\pi(1)}, \dots, w_{\pi(p)})\rho(w_{\pi(p+1)}, \dots, w_{\pi(p+q)}).$$

Assume that $\dim(V) = n$. Then we leave as an exercise the following proposition.

Proposition 7.1. *The direct sum*

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

with the wedge product is an associative algebra.

We call $\Lambda(V^*)$ the *exterior algebra* of V^*. We call an element $\omega \in \Lambda^k(V^*)$ an element of degree k. Because of associativity we can repeatedly wedge and disregard brackets. In particular we can define the wedge product of m elements in V^* and we leave it as an exercise to show that

$$\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^m = \sum_{\pi \in S_m} \mathrm{sgn}(\pi)\alpha^1(v_{\pi(1)})\alpha^2(v_{\pi(2)}) \dots \alpha^m(v_{\pi(m)}).$$

Notice that

$$\alpha^1 \wedge \dots \wedge \alpha^i \wedge \alpha^{i+1} \wedge \dots \wedge \alpha^m = -\alpha^1 \wedge \dots \wedge \alpha^{i+1} \wedge \alpha^i \wedge \dots \wedge \alpha^m$$

and that

$$\alpha^1 \wedge \dots \wedge \alpha \wedge \alpha \wedge \dots \wedge \alpha^m = 0.$$

Still assuming that V is n dimensional choose a basis $v_1, \dots, v_n$ of V. Define the dual basis of V^*, $\alpha^1, \dots, \alpha^n$, by

$$\alpha^i(v_j) = \delta^i_j$$

for all i and j. We want to define a basis of $\Lambda^k(V^*)$. Define elements of $\Lambda^k(V)$ by choosing k numbers $i_1, \dots, i_k$ between 1 and n and considering

$$\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}.$$

As we are trying to form a basis we may as well keep the i_j distinct and ordered $1 \leq i_1 < \dots < i_k \leq n$. We show first that these elements span $\Lambda^k(V^*)$. Let ω be an element of $\Lambda^k(V^*)$. Notice that

$$\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}(v_{j_1}, \dots, v_{j_k})$$

equals zero unless there is a permutation π such that $j_l = i_{\pi(l)}$ for all l and equals $\mathrm{sgn}(\pi)$ if there is such a permutation. Consider vectors $w_1, \dots, w_k$ and expand them in the basis as

$$w_i = \sum_j w_{ij} v_j.$$

Then we have

$$\omega(w_1, \dots, w_k) = \sum_{j_1, \dots, j_k} w_{1j_1} w_{2j_2} \dots w_{kj_k} \omega(v_{j_1}, \dots, v_{j_k})$$

so that it follows that ω is completely determined by its values on basis vectors. For any ordered k-tuple $1 \le i_1 < \cdots < i_k \le n$ define

$$\omega_{i_1 \ldots i_k} = \omega(v_{i_1}, \ldots, v_{i_k})$$

and consider

$$\tilde{\omega} = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

We show that $\omega = \tilde{\omega}$. It suffices to apply both sides to vectors $(v_{i_1}, \ldots, v_{i_k})$ for any $1 \le i_1 < \cdots < i_k \le n$ and show that they are equal but that is clear from previous discussions. So $\Lambda^k(V^*)$ is spanned by the vectors $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$. We have

Proposition 7.2. *The vectors $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$ where $1 \le i_1 < \cdots < i_k \le n$ are a basis for $\Lambda^k(V^*)$.*

Proof. We have already seen that these vectors span. It suffices to show that they are linearly independent. To do this assume that we have

$$0 = \sum_{1 \le i_1 < \cdots < i_k \le n} c_{i_1 \ldots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

If we apply this to $(v_{j_1}, \ldots, v_{j_k})$ we deduce that $c_{j_1 \ldots j_k} = 0$ as required. ∎

It is sometimes useful to sum over all k-tuples $i_1, \ldots, i_k$ not just ordered ones. We can do this—and keep the uniqueness of the coefficients $\omega_{i_1 \ldots i_k}$—if we demand that they be antisymmetric. That is

$$\omega_{j_1 \ldots j_i j_{i+1} \ldots j_k} = -\omega_{j_1 \ldots j_{i+1} j_i \ldots j_k}.$$

Then we have

$$\begin{aligned} \omega &= \textstyle\sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \\ &= \textstyle\sum_{1 \le i_1, \ldots, i_k \le n} \frac{1}{k!} \omega_{i_1 \ldots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}. \end{aligned}$$

We will need one last piece of linear algebra called *contraction*. Let $\omega \in \Lambda^k(V)$ and $v \in V$. Then we define a $k-1$ form $\iota_v \omega$, the contraction of ω and v by

$$\iota_v(\omega)(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1})$$

where $v_1, \ldots, v_{k-1}$ are any $k-1$ elements of V.

Example 7.1. Consider the vector space $\mathbb{R}^3$. Then we know that zero forms and one forms are just real numbers and linear maps respectively. Notice that in the case of $\mathbb{R}^3$ we can identify any linear map v with the vector $v = (v^1, v^2, v^3)$ where

$$v(x) = \sum_{i=1}^{3} v^i x^i.$$

Let α^i be the basis of linear functions defined by $\alpha^i(x) = x^i$. We have seen that every two form ω on $\mathbb{R}^3$ has the form

$$\omega = \omega_1 \alpha^2 \wedge \alpha^3 + \omega_2 \alpha^3 \wedge \alpha^1 + \omega_3 \alpha^1 \wedge \alpha^2.$$

Every three-form μ takes the form

$$\mu = \lambda\ \alpha^1 \wedge \alpha^2 \wedge \alpha^3$$

for some $\lambda \in \mathbb{R}$.

It follows that in $\mathbb{R}^3$ we can identify three-forms with real numbers by identifying μ with λ and we can identify two-forms with vectors by identifying ω with $(\omega_1, \omega_2, \omega_3)$.

It is easy to check that with these identifications the wedge product of two vectors v and w is identified with the vector $v \times w$. In other words wedge product corresponds to cross product.

7.2. **Differential forms and the exterior derivative.** We can now apply the constructions of the previous section to the tangent space to a manifold. We define a k-form on the tangent space at $x \in M$ to be an element of

$$\Lambda^k T_x^* M.$$

We want to define k-form 'fields' in the same way we define vector fields except that we do not call them k-form fields we call them differentiable k-forms or sometimes just k-forms. Choose co-ordinates (U, ψ) on M. Then $\omega(x)$ in $\Lambda^k(T_x^* M)$ can be be written as

$$\omega(x) = \sum_{i_1,\dots,i_k} \frac{1}{k!} \omega_{i_1,\dots,i_k} d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}$$

at each $x \in U$. Hence we have defined a function

$$\omega_{i_1,\dots,i_k} \colon U \to \mathbb{R}$$

for each set of k indices. We call these functions the *components* of ω with respect to the co-ordinate chart. The components satisfy the anti-symmetry conditions in the previous section. We can also define the $\omega_{i_1,\dots,i_k}$ as

$$\omega_{i_1,\dots,i_k} = \omega \left(\frac{\partial}{\partial \psi^{i_1}}, \dots, \frac{\partial}{\partial \psi^{i_k}} \right).$$

We define a smooth differential form by

Definition 7.3 (Differential form)**.** A differential form ω is smooth if its components with respect to a collection of co-ordinate charts whose domains cover M are smooth.

We have the usual Lemma

Lemma 7.4. *If a differential form is smooth then its components with respect to any co-ordinate chart are smooth.*

We denote by $\Omega^k(M)$ the set of all smooth differentiable k forms on M. Notice that $\Omega^0(M)$ is just $C^\infty(M)$ the space of all smooth functions on M.

Exercise 7.1. By analogy with the construction of the tangent bundle show that the kth-exterior power of the dual of the tangent bundle defined by

$$\Lambda^k TM^* = \bigcup_{x \in M} \Lambda^k T_x^* M.$$

can be made into a manifold with a smooth map $\pi \colon \Lambda^k TM^* \to M$ whose smooth sections are precisely the smooth k-forms.

The usual derivative on functions defines a linear differential operator

$$d \colon \Omega^0(M) \to \Omega^1(M).$$

As well as being linear d satisfies the Leibniz rule:

$$d(fg) = f dg + (df)g.$$

We want to show that d extends to act on differential forms of all orders. Before we do that we need two lemmas:

Lemma 7.5. *If $\omega \in \Omega^k(M)$ then for any $x \in M$ there is an open neighbood W of x such that ω restricted to W is a linear combination of differential forms of the form*

$$f_0 df_1 \wedge \dots \wedge df_k$$

for globally defined functions $f_i \colon M \to \mathbb{R}$ for $i = 0, \dots, n$.

Proof. Let (U, ψ) be a co-ordinate chart about x and ω be a k-form which restricted to U is given as

$$\omega|_U = \sum_{i_1 \dots i_k} \frac{1}{k!} \omega_{i_1 \dots i_k} d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}. \tag{7.1}$$

Pick $x \in U$ and extend $\omega_{i_1 \dots i_k}$ and each ψ^i to functions $\widetilde{\omega}_{i_1 \dots i_k}$ and $\widetilde{\psi}^i$ defined on all of M and agreeing with ω and ψ^i on a neighbourhood $W \subseteq U$ of x. Using (7.1) we can construct a global form $\widetilde{\omega}$ agreeing with ω on W

$$\widetilde{\omega} = \sum_{i_1,\dots,i_k} \frac{1}{k!} \widetilde{\omega}_{i_1 \dots i_k} d\widetilde{\psi}^{i_1} \wedge \dots \wedge d\widetilde{\psi}^{i_k}. \tag{7.2}$$

This gives the required result. ∎

and the related

Lemma 7.6. *Let $f_i^a, g_i^a \colon M \to \mathbb{R}$ for $i = 0, 1, \dots, k$, $a = 1, \dots, m$ satisfy*

$$\sum_{a=1}^m f_0^a df_1^a \wedge \dots \wedge df_k^a = \sum_{a=1}^m g_0^a dg_1^a \wedge \dots \wedge dg_k^a$$

then

$$\sum_{a=1}^m df_0^a \wedge df_1^a \wedge \dots \wedge df_k^a = \sum_{a=1}^m dg_0^a \wedge dg_1^a \wedge \dots \wedge dg_k^a.$$

Proof. Consider

$$\begin{aligned} f_0 df_1 \wedge \dots \wedge df_k &= \sum_{i_1,\dots,i_k} f_0 \frac{\partial f_1}{\partial \psi^{i_1}} \cdots \frac{\partial f_k}{\partial \psi^{i_k}} d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k} \\ &= \sum_{i_1,\dots,i_k} \frac{1}{k!} \left(\sum_{\pi \in S_k} \operatorname{sgn}(\pi) f_0 \frac{\partial f_1}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial f_k}{\partial \psi^{i_{\pi(k)}}} \right) d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}. \end{aligned}$$

As the coefficients of the $d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}$ are now anti-symmetric they are determined uniquely so applying this to the equation

$$\sum_{a=1}^m f_0^a df_1^a \wedge \dots \wedge df_k^a = \sum_{a=1}^m g_0^a dg_1^a \wedge \dots \wedge dg_k^a$$

we deduce

$$\sum_{a=1}^m \sum_{\pi \in S_k} \operatorname{sgn}(\pi) f_0^a \frac{\partial f_1^a}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial f_k^a}{\partial \psi^{i_{\pi(k)}}} = \sum_{a=1}^m \sum_{\pi \in S_k} \operatorname{sgn}(\pi) g_0^a \frac{\partial g_1^a}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial g_k^a}{\partial \psi^{i_{\pi(k)}}}.$$

Differentiating the left hand side with respect to ψ^i we obtain

$$\sum_{a=1}^m \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \frac{\partial f_0^a}{\partial \psi^i} \frac{\partial f_1^a}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial f_k^a}{\partial \psi^{i_{\pi(k)}}} + \sum_{a=1}^m \sum_{n=1}^k \sum_{\pi \in S_k} \operatorname{sgn}(\pi) f_0^a \frac{\partial f_1^a}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial^2 f_n^a}{\partial \psi^i \partial \psi^{i_{\pi(m)}}} \cdots \frac{\partial f_k^a}{\partial \psi^{i_{\pi(k)}}}$$

which must equal the corresponding expression with g's. If we multiply this expression by $\frac{1}{k!} d\psi^i \wedge d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}$ and sum the terms involving the symmetric second-order partial derivatives must vanish leaving

$$\sum_{a=1}^m \sum_{i,I} \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \frac{\partial f_0^a}{\partial \psi^i} \frac{\partial f_1^a}{\partial \psi^{i_{\pi(1)}}} \cdots \frac{\partial f_k^a}{\partial \psi^{i_{\pi(k)}}} \frac{1}{k!} d\psi^i \wedge d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k} = \sum_{a=1}^m \sum_{i,I} \frac{\partial f_0^a}{\partial \psi^i} \frac{\partial f_1^a}{\partial \psi^{i_1}} \cdots \frac{\partial f_k^a}{\partial \psi^{i_k}} d\psi^i \wedge d\psi^{i_1} \wedge \dots \wedge d\psi^{i_k}$$

which equals

$$\sum_{a=1}^m df_0^a \wedge df_1^a \wedge \dots \wedge df_k^a.$$

But this must equal the same expression with g's so we have shown that

$$\sum_{a=1}^m df_0^a \wedge df_1^a \wedge \dots \wedge df_k^a = \sum_{a=1}^m dg_0^a \wedge dg_1^a \wedge \dots \wedge dg_k^a.$$

∎

We want to prove

Proposition 7.7. *If the dimension of M is n then there are unique linear maps*

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M)$$

for all $p = 0, \dots, n-1$ satisfying:

1. *If $p = 0$ d is the usual derivative,*
2. *$d^2 = 0$, and*
3. *$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^p \omega \wedge (d\rho)$ where $\omega \in \Omega^p(M)$ and $\rho \in \Omega^q(M)$.*

Proof. Let ω be a differential k-form. Using Lemma 7.5 we can assume that on some open set U we have

$$\omega|_U = \sum_{a=1}^{m} f_0^a df_1^a \wedge \cdots \wedge df_k^a|_U \,. \tag{7.3}$$

So we define

$$d\omega|_U = \sum_{a=1}^{m} df_0^a \wedge df_1^a \wedge \cdots \wedge df_k^a|_U \,. \tag{7.4}$$

To see that $d\omega$ is defined globally we note that from Lemma 7.6 that $d\omega$ as defined is independent of how ω is represented as

$$\omega|_U = \sum_{a=1}^{m} f_0^a df_1^a \wedge \cdots \wedge df_k^a|_U \,.$$

so it will be unique on each open set U on which such a representation exists and thus these definitions will also agree on overlaps. Note that it follows easily from the uniqueness that d is linear.

Notice that $d\omega$ is a global $k+1$-form which has the local form

$$\sum_{a=1}^{m} 1 df_0^a \wedge 1 df_1^a \wedge \cdots \wedge df_k^a.$$

So it follows from the definition that locally $d(d\omega)$ has the form

$$\sum_{a=1}^{m} (d1) \wedge df_0^a \wedge 1 df_1^a \wedge \cdots \wedge df_k^a = 0$$

which proves (2).

Notice also that if $f\colon M \to \mathbb{R}$ then $f\omega$ is a global form that locally has the form

$$\omega|_U = \sum_{a=1}^{m} (f f_0^a) df_1^a \wedge \cdots \wedge df_k^a|_U \,.$$

and thus expanding $d(f f_0^a)$ we see that

$$d\omega = df \wedge \omega + f\omega.$$

As a result we can show that if α and β agree in an open set $U \subseteq M$ then $d\alpha$ and $d\beta$ also agree in U. Indeed by the linearity of d it suffices to show that if $\rho = \alpha - \beta$ vanishes on U so also does $d\rho$. To see this notice that from the discussion of bump functions in Exercise 5.1 if $x \in U$ we can find open sets U_1 and U_2 with $x \in U_1 \subset U_2 \subset U$ and a bump function f on M which vanishes on U_1 and is 1 outside of U_2 so that $\rho = f\rho$ on M. It follows that $d\rho(x) = d(f\rho)(x) = df(x)\rho(x) + f(x)d\rho(x) = 0$. So $d\rho$ vanishes on a neighbourhood of every point in U and hence on U.

It follows now that we only need to prove (3) for forms of the form $\alpha = f_0 df_1 \wedge \cdots \wedge df_p$ and $\beta = g_0 dg_1 \wedge \cdots \wedge dg_q$ and then the general result follows by Lemma 7.5 and linearity. So we have

$$\begin{aligned} d(\alpha \wedge \beta) &= d(f_0 g_0) df_1 \wedge \cdots \wedge df_p \wedge dg_1 \wedge \cdots \wedge dg_q \\ &= (df_0 g_0 + f_0 dg_0) \wedge df_1 \wedge \cdots \wedge df_p \wedge dg_1 \wedge \cdots \wedge dg_q \\ &= df_0 \wedge df_1 \wedge \cdots \wedge df_p \wedge g_0 dg_1 \wedge \cdots \wedge dg_q \\ &\qquad + (-1)^p f_0 \wedge df_1 \wedge \cdots \wedge df_p \wedge dg_0 \wedge dg_1 \wedge \cdots \wedge dg_q \\ &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \end{aligned}$$

Finally we show uniqueness. Note that if we have a linear operator d satisfying the conditions of the theorem then we must have that if $\alpha = \beta$ on an open set U then $d\alpha = d\beta$ on U. Hence if ω has the local form in equation (7.3) then it must be true that $d\omega$ on U is equal to

$$d\left(\sum_{a=1}^{m} f_0^a df_1^a \wedge \cdots \wedge df_k^a\right)$$

But by linearity and (2) and (3) this must give us the right hand side of equation (7.4). So $d\omega$ is determined. □

Definition 7.8. The function

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M)$$

defined in Proposition 7.7 is called the *exterior derivative* (on p-forms).

The condition (3) and others like it is generally called the *Leibniz rule*.

Exercise 7.2. Note that this proposition implies that if

$$\omega = \sum_{i_1,\dots,i_k} \frac{1}{k!} \omega_{i_1\dots i_k} d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}.$$

then we have

$$d\omega = \sum_{i_1,\dots,i_k} \frac{1}{k!} d\omega_{i_1\dots i_k} \wedge d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}$$

or

$$d\omega = \sum_{i_0,i_1,\dots,i_k} \frac{1}{k!} \frac{\partial \omega_{i_1\dots i_k}}{\partial \psi^{i_0}} d\psi^{i_0} \wedge d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}.$$

It is a useful exercise to reprove this formula using this definition of the exterior derivative.

Similarly verify conditions (2) and (3) in Proposition 7.7 on the domain of a co-ordinate chart (U, ψ).

Example 7.2. Recall from 7.1 the way in which we identified one-forms and two-forms on $\mathbb{R}^3$ with vectors. It follows that differentiable one and two forms on $\mathbb{R}^3$ can be identified with vector-fields. Similarly zero and three forms are functions. With these identifications it is straightforward to check that the exterior derivative of zero, one and two forms corresponds to the classical differential operators grad, curl and div.

Exercise 7.3. If X is a vector field and ω is a differential 1-form show that the differential 1-form defined by

$$L_X(\omega) = \sum_{i,j} \left(X^i \frac{\partial \omega_j}{\partial \theta^i} + \omega_i \frac{\partial X^i}{\partial \theta^j}\right) d\theta^j.$$

where

$$X = \sum_i X^i \frac{\partial}{\partial \theta^i} \quad \text{and} \quad \omega = \sum_i \omega_i d\theta^i$$

is actually independent of the choices of co-ordinates. We call $L_X(\omega)$ the Lie derivative of ω by X.

7.3. **Pulling back differential forms.** We have seen that if $f\colon M \to N$ is a smooth map then it has a derivative or tangent map $T_x(f)$ that acts on tangent vectors in T_xM by sending them to $T_{f(x)}N$. Moreover $T_x(f)$ is linear. Recall that if $X\colon V \to W$ is a linear map between vector spaces then it has an adjoint or dual $X^*\colon W^* \to V^*$ defined by

$$X^*(\xi)(v) = \xi(X(v))$$

where $\xi \in W^*$ and $v \in V$. Notice that X^* goes in the opposite direction to X. So we have a linear map called the *cotangent map*

$$T^*_x(f)\colon T^*_{f(x)}N \to T^*_xM$$

which is just the adjoint of the tangent map. It is defined by

$$T^*_x(f)(\omega)(X) = \omega(T_x(f)(X).$$

This action defines a map on differential forms called the *pull-back by* f and denoted f^*. If $\omega \in \Omega^k(N)$ then we define $f^*(\omega) \in \Omega^k(M)$ by

$$f^*(\omega)(x)(X_1, \dots, X_k) = \omega(f(x))(T_x(f)(X_1), \dots, T_x(f)(X_k))$$

for any $X_1, \dots, X_k$ in T_xM.

Notice that if ϕ is a zero form or function on N then $f^*(\phi) = \phi \circ f$. The pull back map

$$f^*\colon \Omega^q(N) \to \Omega^q(M)$$

satisfies the following proposition.

Proposition 7.9. *If $f\colon M \to N$ is a smooth map and ω and μ are a differential forms on N then:*

(1) $df^*(\omega) = f^*(d\omega)$, and
(2) $f^*(\omega \wedge \mu) = f^*(\omega) \wedge f^*(\mu)$.

Proof. Exercise. □

Exercise 7.4. Consider the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. This is a manifold of dimension 1. The circle has a co-ordinate chart (U, θ) where $U = S^1 - \{(1, 0)\}$ and $\theta\colon U \to (0, 2\pi)$ is defined implicitly by

$$(x, y) = (\cos(\theta(x, y)), \sin(\theta(x, y))).$$

That is θ is the usual angle co-ordinate in polar co-ordinates. Identify the tangent space to the circle at (x, y) with the line in $\mathbb{R}^2$ tangential to the circle at (x, y). Calculate a formula for the vector field $\partial/\partial\theta$ in terms of x and y and hence show that it extends from U to a vector field on all of S^1. Show that $d\theta$ also extends to a differential 1-form ω on all of the circle. Show that there is no function $f : S^1 \to \mathbb{R}$ such that $\omega = df$.

Exercise 7.5. Let $S^2 = \{x \in \mathbb{R}^3 | \|x\|^2 = 1\}$ be the two-sphere. Recall that the spherical co-ordinates (θ, ϕ) of the point (x, y, z) on the two-sphere are defined by requiring that:

$$\begin{aligned} x &= \sin(\psi)\cos(\theta) \\ y &= \sin(\psi)\sin(\theta) \\ z &= \cos(\psi). \end{aligned}$$

Find an open set $U \subseteq S^2$ for the domain of the spherical co-ordinates so that $\psi \in (0, \pi)$ and $\theta \in (0, 2\pi)$.

For any x in S^2 and $X, Y \in T_xS^2$ define a differential two-form ω on S^2 by $\omega_x(X, Y) = \langle x, X \times Y\rangle$, where $\langle\ ,\ \rangle$ is the usual inner-product on $\mathbb{R}^3$ and $\times$ is the cross-product of vectors in $\mathbb{R}^3$. By using suitable co-ordinates (spherical are good) calculate the integral of ω over S^2 and show that it is non-zero.

Exercise 7.6. Show that it is not possible to find a differential one-form μ on the two sphere such that $d\mu$ is the volume form ω defined in exercise (7.5).

Exercise 7.7. Consider the torus T^2 in $\mathbb{R}^3$ with co-ordinates (θ, ϕ) defined implicitly by

$$x = (b + a\sin(\phi))\cos(\theta), (b + a\sin(\phi))\sin(\theta), a\cos(\phi)).$$

Calculate $\partial/\partial\psi$ and $\partial/\partial\theta$. Calculate the (outward) unit normal $n(x)$ to the torus, this is the vector in $\mathbb{R}^3$ orthogonal to the tangent space to the torus at x. You will need to draw a picture or something to check it is the outward normal.

Define vol a two-form by $\mathrm{vol}(X,Y) = \langle n, X \times Y\rangle$ and calculate its integral over T^2 when we orient T^2 in such a way as to make vol positive.

7.4. **Integration of differential forms.** Let $U \subseteq \mathbb{R}^n$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then it is well known that if $f\colon \psi(U) \to \mathbb{R}$ is a function then

$$\int_U f \circ \psi \left|\det\left(\frac{\partial \psi^i}{\partial x^j}\right)\right| \, dx^1 \dots dx^n = \int_{\psi(U)} f dx^1 \dots dx^n.$$

In this formula we regard $dx^1 \dots dx^n$ as the symbol for Lebesgue measure. However it is very suggestive of the notation for differential forms developed in the previous section.

If ω is a differential n form on $V = \psi(U)$ then we can write it as

$$\omega(x) = f(x) dx^1 \wedge \dots \wedge dx^n.$$

If we pull it back with the diffeomorphism ψ then, as we seen before,

$$\psi^*(\omega) = f(x) \det\left(\frac{\partial \psi^i}{\partial x^j}\right) dx^1 \wedge \dots \wedge dx^n.$$

So differential n forms transform by the determinant of the jacobian of the diffeomorphism and Lebesgue measure transforms by the absolute value of the determinant of the jacobian of the diffeomorphism. We define the integral of the differential n form ω on $V \subseteq \mathbb{R}^n$ by

$$\int_V \omega = \int_V f(x) dx^1 \dots dx^n$$

when $\omega = f(x)dx^1 \wedge \dots \wedge dx^n$. Alternatively we can write this as

$$\int_V \omega = \int_V \omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) dx^1 \dots dx^n.$$

Call a diffeomorphism $\psi\colon U \to V$ *orientation preserving* if

$$\det\left(\frac{\partial \psi^i}{\partial x^j}(x)\right) > 0$$

for all $x \in U$. Then we have

Proposition 7.10. *If $\psi\colon U \to \psi(U)$ is an orientation preserving diffeomorphism and ω is a differential n form on $\psi(U)$ then*

$$\int_{\psi(U)} \omega = \int_U \psi^*(\omega).$$

We can use this proposition to define the integral of differential forms on a manifold. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ be a covering of M by co-ordinate charts. Choose a partition of unity ϕ_α subordinate to U_α (see Appendix A). Then if ω is a differential n form we can write

$$\omega = \sum_\alpha \phi_\alpha \omega$$

where the support of $\phi_\alpha\omega$ is in U_α. First we define the integral of each of the forms $\phi_\alpha\omega$

$$\int_M \phi_\alpha \omega = \int_{\psi_\alpha(U_\alpha)} (\psi_\alpha^{-1})^*(\phi_\alpha\omega).$$

Then we define the integral of ω to be

$$\int_M \omega = \sum_{\alpha\in I} \int_{\psi_\alpha(U_\alpha)} (\psi_\alpha^{-1})^*(\phi_\alpha\omega).$$

We have to show that this is independent of all the choices we have made. So let us take another open cover $\{(V_\beta, \chi_\beta)\}_{\beta\in J}$ with partition of unity ρ_β. Then we have

$$\begin{aligned}\sum_{\alpha\in I}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*(\phi_\alpha\omega) &= \sum_{\alpha\in I}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*(\sum_{\beta\in J}\rho_\beta\phi_\alpha\omega)\\ &= \sum_{\alpha\in I}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*((\sum_{\beta\in J}\rho_\beta)\phi_\alpha\omega)\\ &= \sum_{\alpha\in I}\sum_{\beta\in J}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*(\rho_\beta\phi_\alpha\omega).\end{aligned}$$

The differential forms $\rho_\beta\phi_\alpha\omega$ have support in $U_\alpha\cap V_\beta$ so we have

$$\begin{aligned}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*(\rho_\beta\phi_\alpha\omega) &= \int_{\psi_\alpha(U_\alpha\cap V_\beta)}(\psi_\alpha^{-1})^*(\rho_\beta\phi_\alpha\omega)\\ &= \int_{\psi_\alpha(U_\alpha\cap V_\beta)}(\psi_\alpha^{-1})^*(\rho_\beta\phi_\alpha\omega)\end{aligned}$$

If the diffeomorphism

$$\chi_\beta\circ\psi_\alpha^{-1}\big|_{\psi(U_\alpha\cap V_\beta)}$$

is orientation preserving then we have

$$\int_{\psi_\alpha(U_\alpha\cap V_\beta)}(\psi_\alpha^{-1})^*(\rho_\beta\phi_\alpha\omega) = \int_{\chi_\beta(U_\alpha\cap V_\beta)}(\chi_\beta^{-1})^*(\rho_\beta\phi_\alpha\omega) = \int_{\chi_\beta(U_\alpha)}(\chi_\beta^{-1})^*(\rho_\beta\phi_\alpha\omega).$$

So we can complete the calculation above and have

$$\sum_{\alpha\in I}\int_{\psi_\alpha(U_\alpha)}(\psi_\alpha^{-1})^*(\phi_\alpha\omega) = \sum_{\beta\in J}\int_{\chi_\beta(U_\beta)}(\chi_\beta^{-1})^*(\rho_\beta\omega).$$

All this calculation rests on the fact that

$$\chi_\beta\circ\psi_\alpha^{-1}\big|_{\psi(U_\alpha\cap V_\beta)}$$

is an orientation preserving diffeomorphism. In general this will not be the case. We have to introduce the notion of an oriented manifold and and oriented co-ordinate chart. Before we can do that we need to discuss orientations on a vector space.

7.5. **Orientation.** Let V be a real vector space of dimension n. Then define $\det(V) = \Lambda^n(V)$. This is a real, one dimensional vector space. So the set $\det(V) = \{0\}$ is *disconnected*. An orientation of the vector space V is a choice of one of these connected components. If X is an invertible linear map from V to V then it induces a linear map from $\det(V)\to\det(V)$ which is therefore multiplication by a complex number. This number is just $\det(X)$ the determinant of X.

If is often useful to think of an orientation in terms of bases of V. If $\mathcal{B}(V)\subset V^n$ denotes the set of all bases of V then it has two connected components. If we have chosen an orientation of V we say that a basis $(v_1,\ldots,v_n)$ is oriented if the n-form $v_1\wedge v_2\wedge\cdots\wedge v_n$ is oriented.

If M is a manifold of dimension n then the same applies to M; $\det(T_xM)-\{0\}$ is a disconnected set. We define

Definition 7.11. A manifold is orientable if there is a non-vanishing n-form on M. Otherwise it is called non-orientable.

Exercise 7.8. Recall the definition of $\mathbb{R}P_2$ the space all lines through the origin in $\mathbb{R}^3$ and its associated co-ordinate charts given in Example 4.12. Calculate the linear relationship between the basis of one forms $d\psi_i^1, d\psi_i^2$ and the basis of one forms $d\psi_j^1, d\psi_j^2$ for $i\neq j$. Hence calculate the relationship between $d\psi_i^1\wedge d\psi_i^2$ and $d\psi_j^1\wedge d\psi_j^2$. Show that $\mathbb{R}P_2$ is not orientable.

Assume that M is connected. If η and ζ are two non-vanishing n forms then $\eta = f\zeta$ for some function f which is either strictly negative or strictly positive. Hence the set of non-vanishing n forms divides into two sets. We have

Definition 7.12 (Orientation)**.** An orientation on M is a choice on each connected component of M of one of these two sets.

An orientation defines an orientation on each tangent space T_xM. We call an n form positive if it coincides with the chosen orientation negative otherwise. We say a chart (U, ψ) is positive or oriented if $d\psi^1 \wedge \cdots \wedge d\psi^n$ is positive. Note that if a chart is not positive we can make it so by changing the sign of one co-ordinate function so oriented charts exits. If we chose two oriented charts then we have that

$$\chi \circ \psi^{-1}\big|_{\psi(U \cap V)}$$

is an oriented diffeomorphism. The converse is also true.

Proposition 7.13. *Assume we have a covering of M by co-ordinate charts $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ such that for any two (U_α, ψ_α) and (U_β, ψ_β) the diffeomorphism*

$$\psi_\beta \circ \psi_\alpha^{-1}\big|_{\psi_\alpha(U_\alpha \cap U_\beta)}$$

is orientation preserving. Then there is an orientation of M which makes each all these charts oriented.

Proof. Notice that the fact that

$$\psi_\beta \circ \psi_\alpha^{-1}\big|_{\psi_\alpha(U_\alpha \cap U_\beta)}$$

is an oriented diffeomorphism means that if $x \in U_\alpha \cap U_\beta$ then

$$d\psi_\alpha^1 \wedge \dots d\psi_\alpha^n(x)$$

is a positive multiple of

$$d\psi_\beta^1 \wedge \dots d\psi_\beta^n(x)$$

Hence if ϕ_α is a partition of unity then

$$\rho = \sum \phi_\alpha d\psi_\alpha^1 \wedge \dots d\psi_\alpha^n(x)$$

is a non-vanishing n form. Clearly this defines the required orientation. ∎

7.6. **Integration again.** We now have the required result that we can integrate differential forms of degree k over a k-dimensional oriented manifold.

8. Stokes theorem.

Recall the Fundamental Theorem of Calculus: If f is a differentiable function then

$$f(b) - f(a) = \int_b^a \frac{df}{dt}(x)dx.$$

In the language we have developed in the previous section this can be written as

$$f(b) - f(a) = \int_{[a,b]} df$$

where we orient the 1-dimensional manifold $[a, b]$ in the positive direction. We want to prove a more general result that will include Stokes theorem, Green's theorem, Gauss' theorem and the Divergence theorem. If M is an oriented manifold of dimension n with boundary ∂M and ω is an $n-1$ form then Stoke's theorem says that

$$\int_M d\omega = \int_{\partial M} \omega.$$

Before we prove this result we need to make sense of the idea of a manifold with boundary.

8.1. **Manifolds with boundary.** We denote by $\mathbb{R}^n_+$ the half-space

$$\mathbb{R}^n_+ = \{(x^1, \ldots, x^n) \mid x^1 \geq 0\}.$$

We define the boundary of $\mathbb{R}^n_+$ to be

$$\partial\mathbb{R}^n_+ = \{(x^1, \ldots, x^n) \mid x^1 = 0\}.$$

and we identify it with $\mathbb{R}^{n-1}$. Recall that a set $U \subseteq \mathbb{R}^n_+$ is open if it is of the form $U = V \cap \mathbb{R}^n_+$ where V is open in $\mathbb{R}^n$. If U is open in $\mathbb{R}^n_+$ we say that $f\colon U \to \mathbb{R}$ is smooth if there is an open set $V \subseteq \mathbb{R}^n$ with $U = V \cap \mathbb{R}^n_+$ and a smooth map $F\colon V \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in U$. If $U \subseteq \mathbb{R}^n_+$ we define $\partial U = U \cap \partial\mathbb{R}^n_+$.

Let M be a set with a subset denoted by ∂M that we call the boundary of M. We say that (U, ψ) is a co-ordinate chart on M if it is a co-ordinate chart as defined before but in addition $\psi(\partial U) \subseteq \partial\psi(U)$, $\psi(\partial U)$ is open in $\partial\psi(U)$,and $\psi_{|\partial U}\colon \partial U \to \partial\psi(U)$ is a bijection. We define compatibility of charts in the usual way but with the extended notion of smoothness above. Once we have this we can define the idea of an atlas and the notion of a manifold M with boundary ∂M. Notice that if we discard the boundary points ∂M we immediately see that $M - \partial M$ is a manifold. Similarly ∂M is a manifold of dimension one less than the dimension of M. We can extend everything we have done so far to the case of manifolds with boundary.

Example 8.1. If M is one-dimensional then T_xM is one-dimensional and an orientation is a choice of a non-zero vector in each T_xM. Intuitively it is like marking an arrow on M indication a direction.

Example 8.2. If M is two-dimensional the intuition is that an orientation is a choice of a direction of rotation. A basis (v_1, v_2) is oriented if v_2 is a positive multiple of a positive rotation of v_1.

Example 8.3. Let Σ be a surface in $\mathbb{R}^3$. Let $n\colon \Sigma \to \mathbb{R}^3$ be a normal vector field. That is $\langle n(x), v\rangle = 0$ for all $v \in T_x\Sigma$. This defines an orientation by declaring $(v, n(x) \times v)$ to be oriented. So for example we can talk about orienting Σ by the outward normal.

If M is a manifold with boundary and $x \in \partial M$ then we can consider $T_x\partial M \subset T_xM$. It makes sense to talk about the vectors in T_xM pointing into M. We say a vetor points out of M if its negative points into M. If M is oriented and $(v_1, \ldots, v_{n-1})$ is an basis of $T_x\partial M$ we define it to be oriented if $(v_0, v_1, \ldots, v_{n-1})$ is oriented where v_0 points out of M. Call this the compatible orientation of ∂M.

Example 8.4. Let $M = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Then M is a manifold with boundary $\partial M = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. Give M the orientation inherited from the usual orientation $((1, 0), (0, 1))$ of $\mathbb{R}^2$. If $(x, y) \in \partial M$ then $((x, y), (-y, x))$ is oriented in M with (x, y) pointing out of M. We can check this by computing

$$(x, y) \wedge (-y, x) = [x(1, 0) + y(0, 1)] \wedge [-y(1, 0) + x(0, 1)] = (1, 0) \wedge (0, 1).$$

Notice that $(-y, x)$ orients the circle ∂M in the usual convention of anticlockwise rotation. So the compatible orientation of the circle is anticlockwise rotation. Notice that this is the usual orientation of the boundary used for Green's Theorem.

8.2. **Stokes theorem.** Let M be a manifold of dimension n with boundary ∂M and let ω be an $n - 1$ form on M with compact support. We want to prove

Theorem 8.1 (Stoke's theorem)**.** *Let M be an oriented manifold with boundary of dimension n and let ω be a differential form of degree $n - 1$ with compact support. If we give ∂M the compatible orientation then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. We cover M with a covering by co-ordinate charts (U_α, ψ_α) and choose a partition of unity ϕ_α subordinate to this cover. Notice that because $\sum_\alpha \phi_\alpha = 1$ we have $\sum_\alpha d\phi_\alpha = 0$ and hence

$$\begin{aligned}\int_M d\omega &= \sum_\alpha \int_M \phi_\alpha d\omega \\ &= \sum_\alpha \int_M d(\phi_\alpha \omega) \\ &= \sum_\alpha \int_{U_\alpha} d(\phi_\alpha \omega) \\ &= \sum_\alpha \int_{\psi_\alpha(U_\alpha)} (\psi^{-1})^* d(\phi_\alpha \omega) \\ &= \sum_\alpha \int_{\psi_\alpha(U_\alpha)} d((\psi^{-1})^*(\phi_\alpha \omega))\end{aligned}$$

and

$$\begin{aligned}\int_{\partial M} \omega &= \sum_\alpha \int_{\partial M} \phi_\alpha \omega \\ &= \sum_\alpha \int_{\partial U_\alpha} \phi_\alpha \omega \\ &= \sum_\alpha \int_{\partial \psi_\alpha(U_\alpha)} (\psi^{-1})^*(\phi_\alpha \omega).\end{aligned}$$

So it suffices prove that

$$\int_{\psi_\alpha(U_\alpha)} d((\psi^{-1})^*(\phi_\alpha \omega)) = \sum_\alpha \int_{\partial \psi_\alpha(U_\alpha)} (\psi^{-1})^*(\phi_\alpha \omega)$$

or equivalently Stoke's theorem for differential forms with compact support on $\mathbb{R}^n_+$. Let us assume then that ω is a differential $(n-1)$-form on U, where U is of the form $U = \mathbb{R}^n_+ \cap V$ for V open in $\mathbb{R}^n$. As as ω is has compact support it has bounded support so there is an $R > 0$ such that if $|x^i| > R$ for all $i = 1, \ldots, n$ then $\omega(x) = 0$. Write $x = (t, y)$ for $y \in \mathbb{R}^{n-1}$ and

$$\omega = \omega_0 \ dy^1 \ldots dy^{n-1} + \sum_{i=1}^{n} \omega_i \ dt \wedge dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^{n-1}$$

so that

$$d\omega = \frac{\partial \omega_0}{\partial t} \ dy^1 \ldots dy^{n-1} + \sum_{i=1}^{n-1} (-1)^i \frac{\partial \omega_i}{\partial y^i} \ dt \wedge dy^1 \wedge \cdots \wedge dy^{n-1}.$$

If $1 \leq j \leq n-1$ then a term of the form

$$\int_U \frac{\partial \omega_j}{\partial y^j} \ dt \wedge dy^1 \wedge \cdots \wedge dy^{n-1}$$

by the Fundamental Theorem of Calculus must be zero as $\omega_j(y^0, \ldots, y^{j-1}, \pm R, y^{j+1}, \ldots, y^{n-1}) = 0$ because of the compact support.

So again by the Fundamental Theorem of Calculus we must have

$$\int_U d\omega = \int_U \frac{\partial \omega_0}{\partial t} \ dy^1 \ldots dy^{n-1} = -\int_{\partial U} \omega(0, y) dy^1 \ldots dy^{n-1} = \int_{\partial U} \omega_{|\partial U}$$

because of our convention on the orientation of boundaries. □

8.3. **Winding numbers.** For use in the next section we need to introduce the concept of winding number. Let $g \colon S^1 \to \mathbb{C} - \{0\}$ be a smooth map. Intuitively as we let $x \in S^1$ move around the circle once $g(x)$ wraps around 0 in $\mathbb{C}$ an integer number of times. This integer is called the winding number of g or $\text{wn}(g)$. A more precised definition can be given as follows.

Proposition 8.2. *Let* $g\colon S^1 \times \mathbb{C}^\times$. *Then*

$$\mathrm{wn}(g) = \frac{1}{2\pi i}\int_{S^1} g^{-1}dg$$

is an integer.

Proof. Parametrise S^1 by $[0, 2\pi]$ and let $g(\theta) = \exp W(\theta)$. Note that $W(2\pi) - W(0) \in 2\pi i\mathbb{Z}$. Then $g^{-1}dg = dW$ and the Fundamental Theorem of Calculus gives the result. ∎

We call $\mathrm{wn}(g)$ the *winding number* of g.

Exercise 8.1. Let $g\colon S^1 \to \mathbb{C}^\times$ be defined by $g(\theta) = \exp(2\pi ik\theta)$. Show that $\mathrm{wn}(g) = k$.

Exercise 8.2. Show that if $g, h\colon S^1 \to \mathbb{C}^\times$ then $\mathrm{wn}(gh) = \mathrm{wn}(g) + \mathrm{wn}(h)$. Similarly show that $\mathrm{wn}(g^{-1}) = -\mathrm{wn}(g)$.

9. Line Bundles

9.1. **Introduction.** The mathematical motivation for studying vector bundles comes from the example of the tangent bundle TM of a manifold M. Recall that the tangent bundle is the union of all the tangent spaces T_mM for every m in M. As such it is a collection of vector spaces, one for every point of M.

The physical motivation comes from the realisation that the fields in physics may not just be maps $\phi : M \to \mathbb{C}^N$ say, but may take values in *different* vector spaces at each point. Tensors do this for example. The argument for this is partly quantum mechanics because, if ϕ is a wave function on a space-time M say, then what we can know about are expectation values, that is things like:

$$\int_M \langle\phi(x), \phi(x)\rangle dx$$

and to define these all we need to know is that $\phi(x)$ takes its values in a one-dimensional complex vector space with Hermitian inner product. There is no reason for this to be the same one-dimensional Hermitian vector space here as on Alpha Centauri. Functions like ϕ, which are generalisations of complex valued functions, are called *sections* of vector bundles.

We will consider first the simplest theory of vector bundles where the vector space is a one-dimensional complex vector space - line bundles.

9.2. **Definition of a line bundle and examples.** The simplest example of a line bundle over a manifold M is the *trivial* bundle $\mathbb{C} \times M$. Here the vector space at each point m is $\mathbb{C} \times \{m\}$ which we regard as a copy of $\mathbb{C}$. The general definition uses this as a local model.

Definition 9.1. A complex line bundle over a manifold M is a manifold L and a smooth surjection $\pi\colon L \to M$ such that:

(1) Each *fibre* $\pi^{-1}(m) = L_m$ is a a complex one-dimensional vector space.
(2) Every $m \in M$ has an open neighbourhood $U \in M$ for which there is a diffeomeorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\varphi(L_m) \subseteq \{m\} \times \mathbb{C}$ for every m and that moreover the map $\varphi|_{L_m} : L_m \to \{m\} \times \mathbb{C}$ is a linear isomorphism.

Note 9.1. The second condition is called *local triviality* because it says that locally the line bundle looks like $\mathbb{C} \times M$. We leave it as an exercise to show that local triviality makes the map π a submersion (that is it has onto derivative) and the scalar multiplication and vector addition maps smooth. In the quantum mechanical example local triviality means that at least in some local region like the laboratory we can identify the Hermitian vector space where the wave function takes its values with $\mathbb{C}$.

Example 9.1. $\mathbb{C} \times M$ the trivial bundle

Example 9.2. Recall that if $u \in S^2$ then the tangent space at u to S^2 is identified with the set $T_u S^2 = \{v \in \mathbb{R}^3 \mid \langle v, u\rangle = 0\}$. We make this two dimensional real vector space a one dimensional complex vector space by defining $(\alpha + i\beta)v = \alpha.v + \beta.u \times v$. We leave it as an exercise for the reader to show that this does indeed make $T_u S^2$ into a complex vector space. What needs to be checked is that $[(\alpha + i\beta)\ (\delta + i\gamma)]v = (\alpha + i\beta)\ [(\delta + i\gamma)]v$ and because $T_u S^2$ is already a real vector space this follows if $i(iv) = -v$. Geometrically this follows from the fact that we have defined multiplication by i to mean rotation through $\pi/2$. We will prove local triviality in a moment.

Example 9.3. If Σ is any surface in $\mathbb{R}^3$ we can use the same construction as in (2). If $x \in \Sigma$ and $\hat{n}_x$ is the unit normal then $T_x\Sigma = \hat{n}_x^{\perp}$. We make this a complex space by defining $(\alpha + i\beta)v = \alpha v + \beta \hat{n}_x \times v$.

Example 9.4 (Hopf bundle). Define $\mathbb{C}P_1$ to be the set of all lines (through the origin) in $\mathbb{C}^2$. Denote the line through the vector $z = (z^0, z^1)$ by $[z] = [z^0, z^1]$. Note that $[\lambda z^0, \lambda z^1] = [z^0, z^1]$ for any non-zero complex number λ. Define two open sets U_i by

$$U_i = \{[z^0, z^1] \mid z^i \neq 0\}$$

and co-ordinates by $\psi_i \colon U_i \to \mathbb{C}$ by $\psi_0([z]) = z^1/z^0$ and $\psi_1([z]) = z^0/z^1$. As a manifold $\mathbb{C}P_1$ is diffeomorphic to S^2. An explicit diffeomorphism $S^2 \to \mathbb{C}P_1$ is given by $(x, y, z) \mapsto [x + iy, 1 - z]$.

We define a line bundle H over $\mathbb{C}P_1$ by $H \subseteq \mathbb{C}^2 \times \mathbb{C}P_1$ where

$$H = \{(w, [z]) \mid w = \lambda z \text{ for some } \lambda \in \mathbb{C}^{\times}\}.$$

We define a projection $\pi \colon H \to \mathbb{C}P_1$ by $\pi((w, [z])) = [z]$. The fibre $H_[z] = \pi^{-1}([z])$ is the set

$$\{(\lambda z, [z]) \mid \lambda \in \mathbb{C}^{\times}\}$$

which is naturally identified with the line through $[z]$. It thereby inherits a vector space structure given by

$$\alpha(w, [z]) + \beta(w', [z]) = (\alpha w + \beta w', [z]).$$

We shall prove later that this is locally trivial.

9.3. **Isomorphism of line bundles.** It is useful to say that two line bundles $L \to M, J \to M$ are isomorphic if there is a diffeomorphism map $\varphi : L \to J$ such that $\varphi(L_m) \subseteq J_m$ for every $m \in M$ and such that the induced map $\varphi|_{L_m} : L_m \to J_m$ is a linear isomorphism.

We define a line bundle L to be *trivial* if it is isomorphic to $M \times \mathbb{C}$ the trivial bundle. Any such isomorphism we call a trivialisation of L.

9.4. **Sections of line bundles.** A section of a line bundle L is like a vector field. That is it is a map $\varphi : M \to L$ such that $\varphi(m) \in L_m$ for all m or more succinctly $\pi \circ \varphi = id_m$.

Example 9.5 (The trivial bundle.). $L = \mathbb{C} \times M$. Every section φ looks like $\varphi(x) = (f(x), x)$ for some function f.

Example 9.6 (The tangent bundle to S^2.). TS^2. Sections are vector fields. Alternatively because each $T_x S^2 \subseteq \mathbb{R}^3$ we can think of a section s as a map $s \colon S^2 \to \mathbb{R}^3$ such that $\langle s(x), x\rangle = 0$ for all $x \in S^2$.

Example 9.7 (The Hopf bundle). By definition a section $s \colon \mathbb{C}P_1 \to H$ is a map

$$s \colon \mathbb{C}P_1 \to H \subseteq \mathbb{C}P_1 \times \mathbb{C}^2$$

which must have the form $[z] \mapsto ([z], w)$. For convenience we will write it as $s([z]) = ([z], s(z))$ where, for any $[z]$ $s \colon \mathbb{C}P_1 \to \mathbb{C}^2$ satisfies $s([z]) = \lambda z$ for some $\lambda \in \mathbb{C}^{\times}$.

The set of all sections, denoted by $\Gamma(M, L)$, is a vector space under pointwise addition and scalar multiplication. I like to think of a line bundle as looking like Figure 1.

Here O is the set of all zero vectors or the image of the *zero section*. The curve s is the image of a section and thus generalises the graph of a function.

We have the following result:

Proposition 9.2. *A line bundle L is trivial if and only if it has a nowhere vanishing section.*

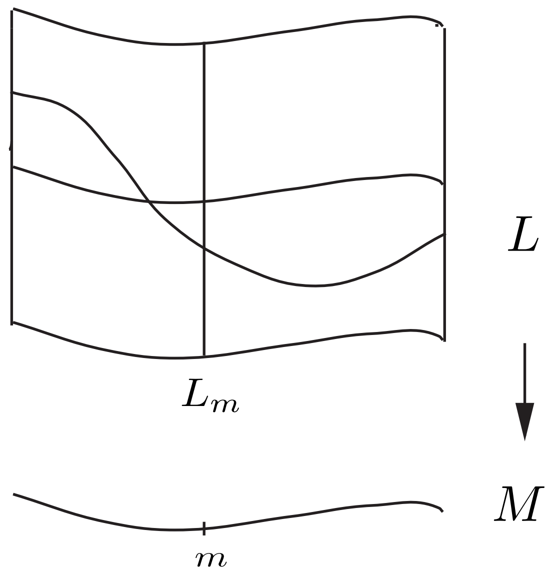

FIGURE 9.1. A line bundle.

Proof. Let $\varphi\colon L \to M \times \mathbb{C}$ be the trivialisation then $\varphi^{-1}(m, 1)$ is a nowhere vanishing section.

Conversely if s is a nowhere vanishing section then define a trivialisation $M \times \mathbb{C} \to L$ by $(m, \lambda) \mapsto \lambda s(m)$. This is an isomorphism. ∎

Note 9.2. . The condition of local triviality in the definition of a line bundle could be replaced by the existence of local nowhere vanishing sections. This shows that TS^2 is locally trivial as it clearly has *local* nowhere-vanishing vector fields. Recall however the so called 'hairy-ball theorem' from topology which tells us that S^2 has no global nowhere vanishing vector fields. Hence TS^2 is not trivial. We shall prove this result a number of times.

9.5. **Transition functions and the clutching construction.** Local triviality means that every property of a line bundle can be understood locally. This is like choosing co-ordinates for a manifold. Given $L \to M$ we cover M with open sets U_α on which there are nowhere vanishing sections s_α. If ξ is a global section of L then it satisfies $\xi|_{U_\alpha} = \xi_\alpha s_\alpha$ for some smooth $\xi_\alpha : U_\alpha \to \mathbb{C}$. The converse is also true. If we can find ξ_α such that $\xi_\alpha s_\alpha = \xi_\beta s_\beta$ for all α, β then they fit together to define a global section ξ with $\xi|_{U_\alpha} = \xi_\alpha s_\alpha$.

It is therefore useful to define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times = \mathbb{C} - \{0\}$ by $s_\alpha = g_{\alpha\beta} s_\beta$. Then a collection of functions ξ_α define a global section if on any intersection $U_\alpha \cap U_\beta$ we have $\xi_\beta = g_{\alpha\beta}\xi_\alpha$. The functions $g_{\alpha\beta}$ are called the *transition functions* of L. We shall see in a moment that they determine L completely. It is easy to show, from their definition, that the transition functions satisfy the condition

$$g_{\beta\gamma}\ g_{\alpha\gamma}^{-1}\ g_{\alpha\beta} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma$$

called the *cocycle condition.*

Exercise 9.1. Show that the cocycle condition is equivalent to the three conditions:

(1) $g_{\alpha\alpha} = 1$

(2) $g_{\alpha\beta} = g_{\beta\alpha}$

(3) $g_{\alpha\beta}\ g_{\beta\gamma}\ g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Proposition 9.3. *Given an open cover $\{U_\alpha\}$ of M and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ satisfying (1) (2) and (3) above we can find a line bundle $L \to M$ with transition functions the $g_{\alpha\beta}$.*

Proof. Consider the disjoint union $\tilde{M}$ of all the U_α. More precisely let I be the indexing set and define $\tilde{M}$ as the subset of $M \times I$ of pairs (m, α) such that $m \in U_\alpha$. There is an obvious map $\tilde{M} \to M$ that sends (m, α) to m. Now

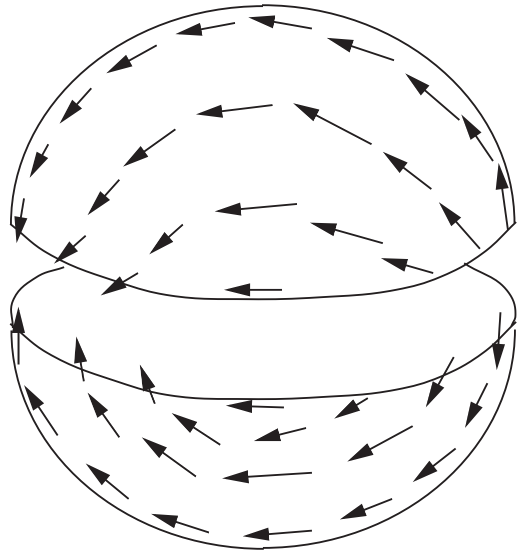

FIGURE 9.2. Vector fields on the two sphere.

consider $\mathbb{C} \times \tilde{M}$ whose elements are triples (λ, m, α) and define $(\lambda, m, \alpha) \sim (\mu, n, \beta)$ if $m = n$ and $g_{\alpha\beta}(m)\lambda = \mu$. We leave it as an exercise to show that $\sim$ is an equivalence relation. Indeed ((1) (2) (3) in the exerice give reflexivity, symmetry and transitivity respectively.)

Denote equivalence classes by square brackets and define L to be the set of equivalence classes. Define addition by $[(\lambda, m, \alpha)] + [(\mu, m, \alpha)] = [(\lambda + \mu, m, \alpha)]$ and scalar multiplication by $z[(\lambda, m, \alpha)] = [(z\lambda, m, \alpha)]$. The projection map is $\pi([(\lambda, m, \alpha)]) = m$. We leave it as an exercise to show that these are all well-defined. Finally define $s_\alpha(m) = [(1, m, \alpha)]$. Then $s_\alpha\ (m) = [(1, m, \alpha)] = [(g_{\alpha\beta}\ (m), m, \beta)] = g_{\alpha\beta}\ (m) s_\beta\ (m)$ as required.

Finally we need to show that L can be made into a differentiable manifold in such a way that it is a line bundle and the s_α are smooth. Denote by W_α the preimage of U_α under the projection map. There is a bijection $\psi_\alpha \colon W_\alpha \to \mathbb{C} \times U_\alpha$ defined by $\psi_\alpha([\alpha, x, z]) = (z, x)$. This is a local trivialisation. If (V, ϕ) is a co-ordinate chart on $U_\alpha \times \mathbb{C}$ then we can define a chart on L by $(\phi_\alpha^{-1}(V), \phi_\alpha \circ \psi_\alpha)$. We leave it as an exercise to check that these charts define an atlas. This depends on the fact that $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ is smooth. □

The construction we have used here is called *the clutching construction*. It follows from this proposition that the transition functions capture all the information contained in L. However they are by no means unique. Even if we leave the cover fixed we could replace each s_α by $h_\alpha s_\alpha$ where $h_\alpha : U_\alpha \to \mathbb{C}^\times$ and then $g_{\alpha\beta}$ becomes $h_\alpha g_{\alpha\beta} h_\beta^{-1}$. If we continued to try and understand this ambiguity and the dependence on the cover we would be forced to invent Cêch cohomology and show that that the isomorphism classes of complex line bundles on M are in bijective correspondence with the Cêch cohomology group $H^1(M, \mathbb{C}^\times)$. We refer the interested reader to [12, 9].

Example 9.8. The tangent bundle to the two-sphere. Cover the two sphere by open sets U_0 and U_1 corresponding to the upper and lower hemispheres but slightly overlapping on the equator. The intersection of U_0 and U_1 looks like an annulus. We can find non-vanishing vector fields s_0 and s_1 as in Figure 2.

If we undo the equator to a straightline and restrict s_0 and s_1 to that we obtain Figure 3.

If we solve the equation $s_0 = g_{01} s_1$ then we are finding out how much we have to rotate s_1 to get s_0 and hence defining the map $g_{01} \colon U_0 \cap U_1 \to \mathbb{C}^\times$ with values in the unit circle. We orient U_0 using the outward normal orientation of S^2 and give the boundary S^1 the induced orientation. Notice that this means that it is oriented from left to right across the page. Inspection of Figure 3 shows that as we go around the equator once s_0 rotates backwards once relative to the usual notion of positive rotation in $\mathbb{C}$ and s_1 rotates forwards once so that thought of as a point on the unit circle in $\mathbb{C}^\times$ g_{01} rotates backwards twice. In other words $g_{01} : U_0 \cap U_1 \to \mathbb{C}^\times$ has winding number -2. This negative two will be important latter.

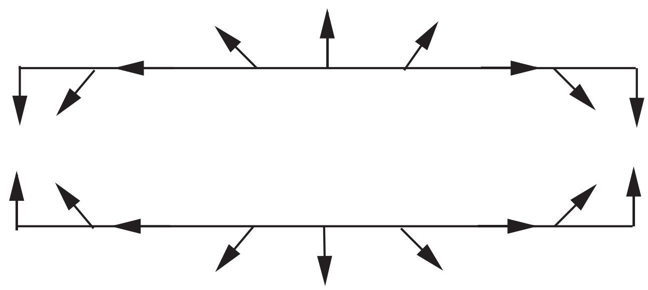

FIGURE 9.3. The sections s_0 and s_1 restricted to the equator.

Example 9.9 (Hopf bundle.). We can define sections $s_i\colon U_i \to H$ for $i = 0, 1$ by

$$s_0[z] = ([z], (1, \frac{z^1}{z^0})) \tag{9.1}$$

$$s_1[z] = ([z], (\frac{z^0}{z^1}, 1)). \tag{9.2}$$

The transition functions are

$$g_{01}([z]) = \frac{z^1}{z^0}.$$

If we consider the circle in U_0 parametrised by $[1, \exp(2\pi i\theta)$ then $g_{01}(\theta) = exp(2\pi i\theta$ has winding number 1.

10. Connections, holonomy and curvature

The physical motivation for connections is that you can't do physics if you can't differentiate the fields! So a connection is a rule for differentiating sections of a line bundle. The important thing to remember is that there is no a priori way of doing this - a connection is a *choice* of how to differentiate. Making that choice is something extra, additional structure above and beyond the line bundle itself. The reason for this is that if $L \to M$ is a line bundle and $\gamma : (-\epsilon, \epsilon) \to M$ a path through $\gamma(0) = m$ say and s a section of L then the conventional definition of the rate of change of s in the direction tangent to γ, that is:

$$\lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

makes no sense as $s(\gamma(t))$ is in the vector space $L_{\gamma(t)}$ and $s(\gamma(0))$ is in the *different* vector space $L_{\gamma(0)}$ so that we cannot perform the required subtraction.

So being pure mathematicians we make a definition by abstracting the notion of derivative:

Definition 10.1. A connection ∇ is a $\mathbb{C}$ linear map

$$\nabla : \Gamma(M, L) \to \Gamma(M, T^*M \otimes L)$$

such that for all s in $\Gamma(M, L)$ and $f \in C^\infty(M, L)$ we have the Liebniz rule:

$$\nabla(fs) = df \otimes s + f \nabla s$$

If $X \in T_xM$ we often use the notation $\nabla_X s = (\nabla s)(X)$.

Example 10.1 (The trivial bundle.). $L = \mathbb{C} \times M$. Then identifying sections with functions we see that (ordinary) differentiation d of functions defines a connection. If ∇ is a general connection then we will see in a moment that $\nabla s - ds$ is a 1-form. So *all* the connections on L are of the form $\nabla = d + A$ for A a 1-form on M (*any* 1-form).

Notice that when we say 1-form here we mean a complex valued 1-form. That is a section of $T^*M \otimes \mathbb{C}$. As any complex valued 1 form takes the form $\alpha + i\beta$ for 1-forms α, β we will avoid introducing this extra notation and just say 1-form and leave it to the reader to interpret if it is supposed to be real or complex from context.

Example 10.2 (The tangent bundle to the sphere.). TS^2. If s is a section then $s : S^2 \to \mathbb{R}^3$ such that $s(u) \in T_uS^2$ that is $\langle s(u), u \rangle = 0$. As $s(u) \in \mathbb{R}^3$ we can differentiate it in $\mathbb{R}^3$ but then ds may not take values in T_uS^2 necessarily. We remedy this by defining

$$\nabla(s) = \pi(ds)$$

where π is orthogonal projection from $\mathbb{R}^3$ onto the tangent space to x. That is $\pi(v) = v - \langle x, v\rangle x$.

Example 10.3 (The tangent bundle to a surface.). A surface Σ in $\mathbb{R}^3$. We can do the same orthogonal projection trick as with the previous example.

Example 10.4 (The Hopf bundle.). Because we have $H \subseteq \mathbb{C}^2 \times \mathbb{C}P_1$ we can apply the same technique as in the previous sections. A section s of H can be identified with a function $s\colon \mathbb{C}P_1 \to \mathbb{C}^2$ such that $s[z] = \lambda z$ for some $\lambda \in \mathbb{C}$. Hence we can differentiate it as a map into $\mathbb{C}^2$. We can then project the result orthogonally using the Hermitian inner product on $\mathbb{C}^2$.

Exercise 10.1. Let ∇^0 and ∇^1 be connections on a complex line bundle L and define

$$\nabla^t(\phi) = t\nabla^1(\phi) + (1-t)\nabla^0(\phi)$$

for any section ϕ of L. Show that ∇^t is a connection for any real number t.

The name connection comes from the name infinitesimal connection which was meant to convey the idea that the connection gives an identification of the fibre at a point and the fibre at a nearby 'infinitesimally close' point. Infinitesimally close points are not something we like very much but we shall see in the next section that we can make sense of the 'integrated' version of this idea in as much as a connection, by parallel transport, defines an identification between fibres at endpoints of a path. However this identification is generally path dependent. Before discussing parallel transport we need to consider two technical points.

The first is the question of existence of connections. We have

Proposition 10.2. *Every line bundle has a connection.*

Proof. Let $L \to M$ be the line bundle. Choose an open covering of M by open sets U_α over which there exist nowhere vanishing sections s_α. If ξ is a section of L write it locally as $\xi_{|U_\alpha} = \xi_\alpha s_\alpha$. Choose a partition of unity ρ_α for subordinate to the cover and note that $\rho_\alpha s_\alpha$ extends to a smooth function on all of M. Then define

$$\nabla(\xi) = \sum_\alpha d\xi_\alpha \rho_\alpha s_\alpha.$$

We leave it as an exercise to check that this defines a connection. ∎

The second point is that we need to be able to restrict a connection to a open set so that we can work with local trivialisations. We have

Proposition 10.3. *If ∇ is a connection on a line bundle $L \to M$ and $U \subseteq M$ is an open set then there is a unique connection ∇_U on $L_{|U} \to U$ satisfying*

$$\nabla(s)_{|U} = \nabla_U(s_{|U}).$$

Proof. We first need to show that if s is a section which is zero in a neighbourhood of a point x then $\nabla(s)(x) = 0$. To show this notice that if s is zero on a neighbourhood U of x then we can find a function ρ on M which is 1 outside U and zero in a neighbourhood of x such that $\rho s = s$. Then we have

$$\nabla(s)(x) = \nabla(\rho s)(x) = d\rho(x)s(x) + \rho(x)\nabla(s)(x) = 0.$$

It follows from linearity that if s and t are equal in a neighbourhood of x then $\nabla(s)(x) = \nabla(t)(x)$. If s is a section of L over U and $x \in U$ then we can multiply it by a bump function which is 1 in a neighbourhood of x so that it extends to a section $\hat{s}$ of L over all of M. Then define $\nabla_U(s)(x) = \nabla(\hat{s})(x)$. If we choose a different bump function to extend s to a different section $\tilde{s}$ then $\tilde{s}$ and $\hat{s}$ agree in a neighbourhood of x so that the definition of $\nabla_U(s)(x)$ does not change. ∎

From now on I will drop the notation $\nabla_{|U}$ and just denote it by ∇.

Let $L \to M$ be a line bundle and $s_\alpha : U_\alpha \to L$ be local nowhere vanishing sections. Define a one-form A_α on U_α by $\nabla s_\alpha = A_\alpha \otimes s_\alpha$. If $\xi \in \Gamma(M, L)$ then $\xi|_{U\alpha} = \xi_\alpha s_\alpha$ where $\xi_\alpha : U_\alpha \to \mathbb{C}$ and

$$\begin{aligned} \nabla \xi|_{U_\alpha} &= d\xi_\alpha s_\alpha + \xi_\alpha \nabla s_\alpha \\ &= (d\xi_\alpha + A_\alpha \xi_\alpha) s_\alpha. \end{aligned} \tag{10.1}$$

Recall that $s_\alpha = g_{\alpha\beta} s_\beta$ for $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C} - \{0\}$ so $\nabla s_\alpha = dg_{\alpha\beta} s_\beta + g_{\alpha\beta} \nabla s_\beta$ and hence $A_\alpha s_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta} s_\alpha + s_\alpha A_\beta$. Hence

$$A_\alpha = A_\beta + g_{\alpha\beta}^{-1} dg_{\alpha\beta} \tag{10.2}$$

on $U_\alpha \cap U_\beta$. The converse is also true. If $\{A_\alpha\}$ is a collection of 1-forms satisfying the equation (10.2) on $U_\alpha \cap U_\beta$ then there is a connection ∇ such that $\nabla s_\alpha = A_\alpha s_\alpha$. The proof is an exercise using equation (10.1) to define the connection.

10.1. **Parallel transport and holonomy.** If $\gamma : [0, 1] \to M$ is a path and ∇ a connection we can consider the notion of moving a vector in $L_{\gamma(0)}$ to $L_{\gamma(1)}$ without changing it, that is *parallel transporting* a vector from $L_{\gamma(0)}, L_{\gamma(1)}$. Here change is measured relative to ∇ so if $\xi(t) \in L_{\gamma(t)}$ is moving without changing it must satisfy the differential equation:

$$\nabla_{\dot\gamma} \xi = 0 \tag{10.3}$$

where $\dot\gamma$ is the tangent vector field to the curve γ. Assume for the moment that the image of γ is inside an open set U_α over which L has a nowhere vanishing section s_α. Then using (10.3) and letting $\xi(t) = \xi_\alpha(t) s_\alpha(\gamma(t))$ we deduce that

$$\frac{d\xi_\alpha}{dt} = -A_\alpha(\gamma)\xi_\alpha$$

or

$$\xi_\alpha(t) = \exp\big(-\int_0^t A_\alpha(\gamma(t)\big)\xi_\alpha(0) \tag{10.4}$$

This is an ordinary differential equation so standard existence and uniqueness theorems tell us that parallel transport defines an isomorphism $L_{\gamma(0)} \to L_{\gamma(t)}$. Moreover if we choose a curve not inside a special open set like U_α we can still cover it by such open sets and deduce that the parallel transport

$$P_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$$

is an isomorphism. In general P_γ is dependent on γ and ∇. The most notable example is to take γ a *loop* that is $\gamma(0) = \gamma(1)$. Then we define $\mathrm{hol}(\gamma, \nabla)$, the *holonomy* of the connection ∇ around the loop γ by taking any $s \in L_{\gamma(0)}$ and defining

$$P_\gamma(s) = \mathrm{hol}\ (\gamma, \nabla).s$$

Example 10.5. A little thought shows that ∇ on the two sphere preserves lengths and angles, it corresponds to moving a vector so that the rate of change is normal. If we consider the 'loop' in Figure 4 then we have drawn parallel transport of a vector and the holonomy is $\exp(i\theta)$.

10.2. **Curvature.** If we have a loop γ whose image is in U_α then we can apply (10.4) to obtain

$$\mathrm{hol}\ (\nabla, \gamma) = \exp\ (-\int_\gamma A_\alpha).$$

If γ is the boundary of a disk D then by Stokes' theorem we have

$$\mathrm{hol}\ (\nabla, \gamma) = \exp - \int_D dA_\alpha. \tag{10.5}$$

Consider the two-forms dA_α. From (10.2) we have

$$\begin{aligned} dA_\alpha &= dA_\beta + d\left(g_{\alpha\beta}^{-1} dg_{\alpha\beta}\right) \\ &= dA_\beta - g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} ddg_{\alpha\beta} \\ &= dA_\beta. \end{aligned}$$

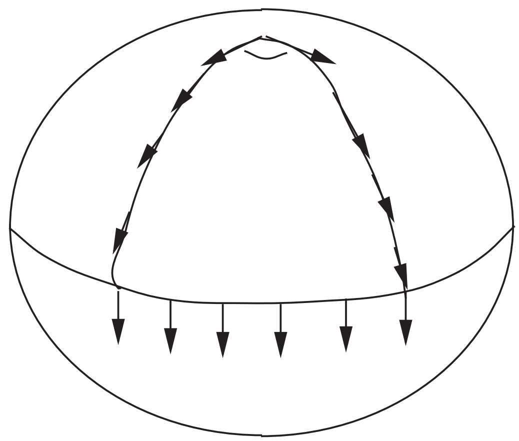

FIGURE 10.1. Parallel transport on the two sphere.

Here we use the useful fact that if $h\colon M \to \mathbb{C} - \{0\}$ then $d(h^{-1}) = -h^{-2}dh$ which follows from differentiating $h^{-1}h = 1$. So the two-forms dA_α agree on the intersections of the open sets in the cover and hence define a *global* two form that we denote by F and call the *curvature* of ∇. Then we have

Proposition 10.4. *If $L \to M$ is a line bundle with connection ∇ and Σ is a compact submanifold of M with boundary a loop γ then*

$$\text{hol }(\nabla, \gamma) = \exp\ -\int_\Sigma F$$

Proof. Notice that (10.5) gives the required result if Σ is a disk which is inside one of the U_α. Now consider a general Σ. By compactness we can triangulate Σ in such a way that each of the triangles is in some U_α. Now we can apply (10.5) to each triangle and note that the holonomy up and down the interior edges cancels to give the required result. ∎

Example 10.6. We calculate the holonomy of the standard connection on the tangent bundle of S^2. Let us use polar co-ordinates: The co-ordinate tangent vectors are:

$$\begin{aligned}
\frac{\partial}{\partial\theta} &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\
\frac{\partial}{\partial\phi} &= (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi))
\end{aligned}$$

Taking the cross product of these and normalising gives the unit normal

$$\begin{aligned}
\hat{n} &= (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \\
&= \sin(\phi)\frac{\partial}{\partial\phi} \times \frac{\partial}{\partial\theta}
\end{aligned}$$

To calculate the connection we need a non-vanishing section s we take

$$s = (-\sin(\theta), \cos(\theta), 0)$$

and then

$$ds = (-\cos(\theta), -\sin(\theta), 0)d\theta$$

so that

$$\begin{aligned}\nabla s &= \pi(ds) \\ &= ds - < ds, \hat{n} > \hat{n} \\ &= (-\cos(\theta), -\sin(\theta), 0)d\theta \\ &\quad + \sin(\phi)\ (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))d\theta \\ &= (-\cos(\theta)\cos^2(\phi), -\sin(\theta)\cos^2(\phi), \cos(\phi)\sin(\phi))d\theta \\ &= \cos(\phi)(\hat{n}\times s)d\theta \\ &= [i\cos(\phi)d\theta]s\end{aligned}$$

Hence $A = i\cos(\phi)d\theta$ and $F = i\sin(\phi)d\theta \wedge d\phi$. To understand what this two form is note that the volume form on the two-sphere is $\mathrm{vol} = -\sin(\phi)d\theta \wedge d\phi$ and hence $F = -i\mathrm{vol}$ The region bounded by the path in Figure 4 has area θ. If we call that region D we conclude that

$$\exp(-\int_D F) = \exp i\theta.$$

Note that this agrees with the previous calculation in Example 10.5 for the holonomy around this particular path.

10.3. **Curvature as infinitesimal holonomy.** The equation $\mathrm{hol}(-\nabla, \partial D) = \exp\ (-\int_D F)$ has an infinitesimal counterpart. If X and Y are two tangent vectors and we let D_t be a parallelogram with sides tX and tY then the holonomy around D_t can be expanded in powers of t as

$$\mathrm{hol}\ (\nabla, D_t) = 1 + t^2\ F(X,Y) + 0(t^3).$$

11. Chern numbers

In this section we define the Chern number which is a (topological) invariant of a line bundle. Before doing this we collect some facts about the curvature.

Proposition 11.1. *The curvature F of a connection ∇ satisfies:*

(a) $dF = 0$
(b) If ∇, ∇' are two connections then $\nabla = \nabla' + \eta$ for η a 1-form and $F_\nabla = F_{\nabla'} + d\eta$.
(c) If Σ is a closed surface then $\frac{1}{2\pi i}\int_\Sigma F_\nabla$ is an integer independent of ∇.

Proof. (a) $dF|_{U\alpha} = d(dA_\alpha) = 0$.

(b) Locally $A'_\alpha = A_\alpha + \eta_\alpha$ as $\eta_\alpha = A'_\alpha - A_\alpha$. But $A_\beta = A_\alpha - g^{-1}_{\alpha\beta}dg_{\alpha\beta}$ and $A'_\beta = A'_\alpha - g^{-1}_{\alpha\beta}dg_{\alpha\beta}$ so that $\eta_\beta = \eta_\alpha$. Hence η is a global 1-form and $F_\nabla = dA_\alpha$ so $F'_\nabla = F_\nabla + d\eta$.

(c) If Σ is a closed surface then $\partial\Sigma = \emptyset$ so by Stokes' theorem $\int_\Sigma F_\nabla = \int_\Sigma F'_\nabla$. Now choose a family of disks D_t in Σ whose limit as $t \to 0$ is a point. For any t the holonomy of the connection around the boundary of D_t can be calculated by integrating the curvature over D_t or over the complement of D_t in Σ and using Proposition 2.1. Taking into account orientation this gives us

$$\exp(\int_{\Sigma - D_t} F) = \exp(-\int_{D_t} F)$$

and taking the limit as $t \to 0$ gives

$$\exp(\int_\Sigma F) = 1$$

which gives the required result. ∎

The Chern number, $c(L)$, of a line bundle $L \to \Sigma$ where Σ is an oriented surface is defined to be the integer $\frac{1}{2\pi i}\int_\Sigma F_\nabla$ for any connection ∇.

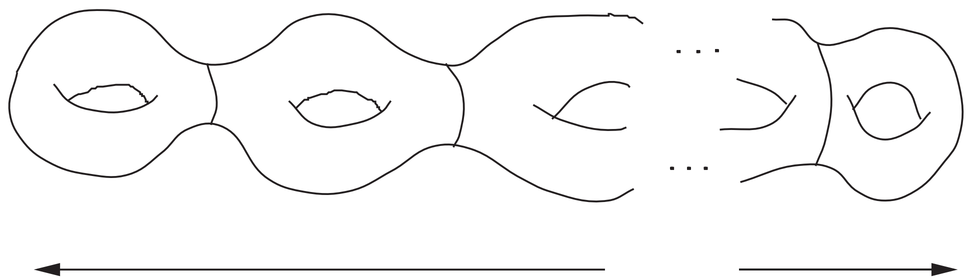

Figure 11.1. A surface of genus g.

Exercise 11.1. Show that if $L \to M$ is a trivial bundle then it has zero Chern number.

Exercise 11.2. Consider the Hopf bundle H over $\mathbb{C}P_1$. Define parameters on $U_0 = \mathbb{C}P_1 - [1,0]$ by $(x,y) \mapsto [x+iy,1]$. Let $s_0([x+iy,1]) = ([x+iy,1],(x+iy,1))$ be the section defined in class. Using (hermitian) orthogonal projection define a connection ∇ on H and calculate the connection one form A_0. Be careful to make the orthogonal projection complex linear. Calculate the curvature over the open set U_0 and integrate it over U_0 to find the Chern number of H. You may find it convenient to work with the complex differential forms $dz = dx + idy$ and $d\bar{z} = dx - idy$.

Exercise 11.3. Consider the tangent bundle to the two-sphere. Give it the connection defined by orthogonal projection and calculate its curvature and hence the Chern number of the tangent bundle to the two-sphere.

Exercise 11.4. Repeat Exercise 11.3 for the torus using the co-ordinates defined in Exercise 7.7.

Exercise 11.5. This assumes you are familiar with the Gauss-Bonnet theorem. If Σ is a closed surface in $\mathbb{R}^3$ define a connection on its tangent bundle by using orthogonal projection. Relate the curvature of this connection to the usual Gaussian curvature.

Example 11.1. For the case of the two sphere Example 10.6 showed that $F = -i\,\mathrm{vol}_{S^2}$. Hence

$$c(TS^2) = \frac{-i}{2\pi i}\int_{S^2} \mathrm{vol} = \frac{-i}{2\pi i}\,4\pi = -2.$$

Some further insight into the Chern number can be obtained by considering a covering of S^2 by two open sets U_0, U_1 as in Figure 2. Let $L \to S^2$ be given by a transition for $g_{01} : U_0 \cap U_1 \to \mathbb{C}^\times$. Then a connection is a pair of 1-forms A_0, A_1, on U_0, U_1 respectively, such that

$$A_1 = A_0 + dg_{10}g_{10}^{-1} \text{ on } U_0 \cap U_1.$$

Take $A_1 = 0$ and A_0 to be any extension of $dg_{01}g_{01}^{-1}$ to U_0. Such an extension can be made by shrinking U_0 and U_1 a little and using a cut-off function. Then $F = dA_1 = 0$ on U_1 and $F = dA_0$ on U_0. To find $c(L)$ we note that by Stokes theorem:

$$\int_{S^2} F = \int_{U_0} F = \int_{\partial U_0} A_0 = \int_{\partial U_1} g_{01}^{-1} dg_{01}.$$

so that

$$c(TS^2) = \frac{1}{2\pi i}\int_{S^2} F = \mathrm{wn}(g_{01})$$

Note that we have already seen for TS^2 in Examples 9.8 and 11.1 that the winding number of g_{01} and the Chern number of the tangent bundle are both -2. It is not difficult to go further now and prove that isomorphism classes of line bundles on S^2 are in one to one correspondence with the integers via the Chern number but will not do this here.

Example 11.2. Another example is a surface Σ_g of genus g as in Figure 5. We cover it with g open sets $U_1, \dots, U_g$ as indicated. Each of these open sets is diffeomorphic to either a torus with a disk removed or a torus with two disks removed. A torus has a non-vanishing vector field on it. If we imagine a rotating bicycle wheel then the inner tube of the tyre (ignoring the valve!) is a torus and the tangent vector field generated by the rotation defines a non-vanishing vector field. Hence the same is true of the open sets in Figure 5. There are corresponding transition functions $g_{12}, g_{23}, \dots, g_{g-1g}$ and we can define a connection in a manner analogous to the two-sphere case and we find that

$$c(T\Sigma_g) = \sum_{i=1}^{g-1} \mathrm{wn}(g_{i,i+1}).$$

All the transition functions have winding number 2 so that

$$c(T\Sigma_g) = 2g - 2.$$

This is a form of the Gauss-Bonnet theorem. It would be a good exercise for the reader familiar with the classical Riemannian geometry of surfaces in $\mathbb{R}^3$ to relate this result to the Gauss-Bonnet theorem. In the classical Gauss-Bonnet theorem we integrate the Gaussian curvature which is the trace of the curvature of the Levi-Civita connection.

So far we have only defined the Chern number for a surface. To define it for manifolds of higher dimension we need the notion of de Rham cohomology [4] which we consider next.

12. De Rham cohomology

The rest of these notes lack detailed proofs. They can all be found in [4].

12.1. **Introduction to de Rham cohomology.**

Definition 12.1. If M is a manifold of dimension n we define the *de Rham complex* to be the sequence of spaces

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

where $\Omega^p(M)$ is the space of all p forms on M with the horizontal maps d the exterior derivatives.

Definition 12.2. The *de Rham cohomology* spaces of a manifold M are defined by

$$H^q(M) = \frac{\ker d\colon \Omega^q(M) \to \Omega^{q+1}(M)}{\operatorname{image} d\colon \Omega^{q-1}(M) \to \Omega^q(M)}.$$

Example 12.1. $H^*(\mathbb{R})$, $H^*((a,b))$.

Example 12.2 (Winding class). Notice that if $g\colon M \to \mathbb{C}^\times$ we can define $\operatorname{wc}(g) = \frac{1}{2\pi i}[g^{-1}dg] \in H^1(M)$ the winding class of g. We shall see that integration defines an isomorphism

$$\begin{aligned} H^1(S^1) &\to \mathbb{R} \\ [\omega] &\mapsto \int_M \omega \end{aligned}$$

so that if M is the circle then we can identify $\operatorname{wc}(g) = \operatorname{wn}(g)$.

Example 12.3 (Chern number). The general definition of $c(L)$ is to take the de Rham cohomology class in $H^2(M)$ containing $\frac{1}{2\pi i}F_\nabla$ for some connection. That this is well-defined follows from Proposition 11.1.

It is a standard result [4] that if M is oriented, compact, connected and two dimensional integrating representatives of degree two cohomology classes defines an isomorphism

$$\begin{aligned} H^2(M) &\to \mathbb{R} \\ [\omega] &\mapsto \int_M \omega \end{aligned}$$

where $[\omega]$ is a cohomology class with representative form ω. Hence we recover the definition of the Chern number of a line bundle over an oriented surface.

12.2. **Some homological algebra.**

Definition 12.3. A sequence of vector spaces and linear maps

$$A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^3 \xrightarrow{d^3} \cdots \xrightarrow{d^{n-1}} A^n$$

is called a *(cochain) complex* if $d^{i+1} \circ d^i = 0$ for all i. We denote it by $A^\bullet$ and typically suppress the label on the maps d which are called *differentials.*

Note 12.1. A more general definition of cochain complex is to take the spaces to be abelian groups or modules and the differentials to be homomorphisms. Many of the constructions we give will work in that case as well.

Note 12.2. If we reversed the direction of all the differentials we would obtain the definition of a chain complex.

Definition 12.4. The *cohomology* of a cochain complex is defined by

$$H^q(A^\bullet) = \frac{\ker d\colon A^q \to A^{q+1}}{\operatorname{image} d\colon A^{q-1} \to A^q}$$

for all q with the end cases being

$$H^0(A^\bullet) = \ker d\colon A^0 \to A^1$$

and

$$H^n(A^\bullet) = \frac{A^n}{\operatorname{image} d\colon A^{n-1} \to A^n}$$

Note 12.3. Notice that we don't need to worry about making a separate definition at the endpoints if we define $\cdots = A^{-2} = A^{-1} = 0 = A^{n+1} = A^{n+2} = \cdots$. Notice all that the only linear maps $0 \to A$ and $A \to 0$ are the zero map.

Definition 12.5. A sequence of vector spaces and linear maps

$$V^1 \xrightarrow{f^1} V^2 \xrightarrow{f^2} V^3 \xrightarrow{f^3} \cdots \xrightarrow{f^{n-1}} V^n$$

is called *exact at* V_i if the image of $f^{i-1}\colon V^{i-1} \to V^i$ equals the kernel of $f^i\colon V^i \to V^{i+1}$. We say it is *exact* if it is exact at V_i for all i.

Proposition 12.6. *(1) The sequence $0 \to V \xrightarrow{\alpha} W$ is exact at V if and only if α is injective.*

(2) The sequence $V \xrightarrow{\alpha} W \to 0$ is exact at W if and only if α is surjective.

(3) The sequence $0 \to V \xrightarrow{\alpha} W \to 0$ is exact if and only if α is an isomorphism.

Definition 12.7. A sequence

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$$

which is exact is called a *short exact sequence.*

Note 12.4. For a short exact sequence we must have α injective and β surjective and thus $V/\alpha(U) \simeq W$. If U, V and W are finite dimensional then $\dim(V) = \dim(U) + \dim(W)$.

Proposition 12.8. *If*

$$0 \xrightarrow{f_0} V^1 \xrightarrow{f_1} V^2 \xrightarrow{f_2} V^3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} V^n \xrightarrow{0} 0$$

is a long exact sequence

$$\sum_i (-1)^i \dim V^i = 0.$$

Definition 12.9. If $A^\bullet$ and $B^\bullet$ are complexes then a *chain map* $f^\bullet\colon A^\bullet \to B^\bullet$ is a collection of linear maps $f^i\colon A^i \to B^i$ with the property that all the diagrams of the form

$$\begin{array}{ccc} A^i & \xrightarrow{f^i} & B^i \\ {\scriptstyle d^i}\big\downarrow & & \big\downarrow{\scriptstyle d^i} \\ A^{i+1} & \xrightarrow[f^{i+1}]{} & B^{i+1} \end{array}$$

commute.

Example 12.4. If $f\colon N \to M$ then f^* is a cochain map from $\Omega^\bullet(N) \to \Omega^\bullet(M)$.

Proposition 12.10. *If $f^\bullet\colon A^\bullet \to B^\bullet$ is a chain map there is a well-defined map*

$$H^q(f^\bullet)\colon H^q(A^\bullet) \to H^q(B^\bullet)$$

given by $H^q(f)([a]) = [f^q(a)]$.

Example 12.5. If $f\colon N \to M$ then f^* is a cochain map from $\Omega^\bullet(N) \to \Omega^\bullet(M)$ and hence induces a map $H^q(f^*)\colon H^q(M) \to H^q(N)$.

Definition 12.11. A sequence of cochain complexes

$$\cdots \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to \cdots$$

is exact at $B^\bullet$ if each

$$\cdots \to A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i \to \cdots$$

is exact at B^i for all i. Similarly a sequence of cochain complexes is exact if it is exact at every term.

Proposition 12.12. *If*

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

is a short exact sequence of complexes then there is a long exact sequence in cohomology

$$\begin{array}{lllllll} 0 \longrightarrow & H^0(A^\bullet) & \xrightarrow{H^0(f^\bullet)} & H^0(B^\bullet) & \xrightarrow{H^0(g^\bullet)} & H^0(C^\bullet) & \xrightarrow{\partial} \\ \longrightarrow & H^1(A^\bullet) & \xrightarrow{H^1(f^\bullet)} & H^1(B^\bullet) & \xrightarrow{H^1(g^\bullet)} & H^1(C^\bullet) & \xrightarrow{\partial} \\ \longrightarrow & H^2(A^\bullet) & \xrightarrow{H^2(f^\bullet)} & \cdots\cdots & & & . \end{array} \tag{12.1}$$

Note 12.5. The map $\partial\colon H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ is called the *connecting homomorphism.*

12.3. Mayer-Vietoris sequence.

Lemma 12.13. *Let $M = U \cup V$ where U and V are open. Then if α_U and α_V are differential q forms on U and V which agree on restriction to $U \cap V$ then there is a differential q form α on M with $\alpha|_U = \alpha_U$ and $\alpha|_V = \alpha_V$.*

Proposition 12.14. *Let $M = U \cup V$ where U and V are open. Then the sequence of complexes*

$$0 \to \Omega^q(M) \to \Omega^q(U) \oplus \Omega^q(V) \to \Omega^q(U \cap V) \to 0$$

is exact. The first map here is $\omega \mapsto (\omega|_U, \omega|_V)$ and the second map is $(\alpha, \beta) \mapsto \alpha|_{U\cap V} - \beta|_{U \cap V}$.

Proposition 12.15 (Mayer-Vietoris)**.** *Let $M = U \cup V$ where U and V are open. Then there is a long exact sequence of cohomology groups*

$$\begin{array}{llllllll} 0 \longrightarrow & H^0(M) & \longrightarrow & H^0(U) \oplus H^0(V) & \longrightarrow & H^0(U\cap V) & & \\ \longrightarrow & H^1(M) & \longrightarrow & H^1(U) \oplus H^0(V) & \longrightarrow & H^1(U\cap V) & & \\ \longrightarrow & H^2(M) & \longrightarrow & H^2(U) \oplus H^0(V) & \longrightarrow & H^2(U\cap V) & \longrightarrow & \cdots . \end{array} \tag{12.2}$$

Example 12.6. $H^0(S^1) = \mathbb{R}$ and $H^1(S^1) = \mathbb{R}$.

Definition 12.16 (Euler characteristic)**.** If M is a manifold of dimension n we define by the *Euler characteristic* of M by

$$\chi(M) = \sum_{q=0}^{n} \dim H^q(M).$$

Proposition 12.17. *If $M = U \cup V$ where U and V are open then $\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$.*

Definition 12.18. We say that cochain maps $f^\bullet, g^\bullet\colon A^\bullet \to B^\bullet$ are *cochain homotopic* if there exists a *(cochain) homotopy operator* which is a family of maps $K\colon A^q \to B^{q-1}$ satisfying

$$f - g = \pm dK \pm Kd.$$

Note 12.6. The signs in this formula can be independent of each other and the following proposition still holds.

Proposition 12.19. *If $f^\bullet, g^\bullet\colon A^\bullet \to B^\bullet$ are cochain homotopic then $H^q(f^\bullet) = H^q(g^\bullet)$ for all q.*

12.4. **Poincare Lemma.**

Theorem 12.20. *If $\pi\colon M\times\mathbb{R}\to M$ is the projection map $\pi(m,t)=m$ then*

$$H^q(\pi^*)\colon H^q(M)\to H^q(M\times\mathbb{R})$$

is an isomorphism. The inverse is given by s^* where $s\colon M\to M\times\mathbb{R}$ is any section of the form $s(m)=(m,t)$ for fixed t.

Proposition 12.21 (Poincare Lemma)**.**

$$H^q(\mathbb{R}^n)=\begin{cases}\mathbb{R} & q=0\\ 0 & 0<q\le n.\end{cases}$$

Definition 12.22. Two smooth maps f and g from M to N are homotopic if there is a smooth map $F\colon M\times\mathbb{R}\to N$ such that for all $m\in M$ we have $F(m,t)=f(m)$ for $t\le 0$ and $F(m,t)=g(m)$ for $t\ge 1$.

Theorem 12.23 (Homotopy axiom)**.** *If $f,g\colon M\to N$ are homotopic then $H^q(f^*)=H^q(g^*)$ for all q.*

Example 12.7.

$$H^q(S^n)=\begin{cases}\mathbb{R} & q=0\\ 0 & 0<q<n\\ \mathbb{R} & q=n.\end{cases}$$

Example 12.8. Adding handles to a surface.

Proposition 12.24 (Brouwer Fixed Point Theorem)**.** *Let $B\subset\mathbb{R}^n$ be the unit ball. If $f\colon B\to B$ is smooth then it has a fixed point. That is there is some $x\in B$ such that $f(x)=x$.*

Definition 12.25. A manifold M is called *contractible* if the identity map is homotopic to some constant map $M\to M$ of the form $m\mapsto m_0$ for a fixed $m_0\in M$.

Example 12.9. $\mathbb{R}^n$ is contractible

Proposition 12.26. *If M is contractible then*

$$H^q(M)=\begin{cases}\mathbb{R} & q=0\\ 0 & 0<q\end{cases}$$

Proposition 12.27. *Let $\pi\colon L\to M$ be a line bundle. Then $\pi^*\colon H^q(M)\to H^q(L)$ is an isomorphism for all q.*

Proposition 12.28.

$$H^q(\mathbb{C}P_2)=\begin{cases}\mathbb{R} & q=0,2,4\\ 0 & 0=1,3\end{cases}$$

Note 12.7. More generally the same method shows that the cohomology of $\mathbb{C}P_n$ is $\mathbb{R}$ in even dimension and 0 in odd dimension.

13. The Mayer-Vietoris argument

Definition 13.1. A cover $\{U_\alpha\}_{\alpha\in I}$ of a manifold M is called a *good cover* if every finite intersection $U_{\alpha_1}\cap U_{\alpha_2}\cap\cdots\cap U_{\alpha_r}$ is contractible.

Proposition 13.2. *Every manifold has a good cover. A compact manifold has a finite good cover.*

13.1. **Finite dimensionality of de Rham cohomology.**

Proposition 13.3. *If M has a finite good cover all its cohomology groups are finite dimensional.*

13.2. **Poincare duality.**

Definition 13.4. We say that $\omega \in \Omega^p(M)$ is *compactly supported* if $\mathrm{supp}(\omega)$ is compact. We denote the compactly supported forms by $\Omega^p_c(M)$.

Note 13.1. Notice that $d(\Omega^p_c(M)) \subset \Omega^{p+1}_c(M)$.

Definition 13.5. The cohomology of $\Omega^\bullet_c(M)$ is called the *compactly supported cohomology* of M and denoted $H^q_c(M)$.

Note 13.2. If $\omega \in \Omega^q(U)$ is compactly supported for U open in M then $\mathrm{supp}(\omega)$ is closed in M so we can extend ω by zero to a form $\mathrm{ext}_M(\omega) \in \Omega^q(M)$.

Proposition 13.6. *Let $M = U \cup V$ where U and V are open. Then the sequence of complexes*

$$0 \to \Omega^q_c(U \cap V) \to \Omega^q_c(U) \oplus \Omega^q_c(V) \to \Omega^q_c(M) \to 0$$

is exact. The first map here is $\omega \mapsto (\mathrm{ext}_U(\omega), \mathrm{ext}_V(\omega))$ and the second map is $(\alpha, \beta) \mapsto \mathrm{ext}_M(\alpha) - \mathrm{ext}_M(\beta))$.

Note 13.3. There is a corresponding long exact Mayer-Vietoris sequence for compactly supported cohomology.

Proposition 13.7. *If M is a manifold then $H^q_c(M \times \mathbb{R}) = H^{q-1}_c(M)$.*

Proposition 13.8 (Poincare Lemma for compactly supported cohomology)**.**

$$H^q_c(\mathbb{R}^n) = \begin{cases} 0 & 0 \leq q < n \\ \mathbb{R} & q = n \end{cases}$$

Note 13.4. The isomorphism $H^n_c(\mathbb{R}^n) \to \mathbb{R}$ is given by integration.

Proposition 13.9. *If M has a finite good cover its compactly supported cohomology is finite-dimensional.*

Proposition 13.10 (Poincare Duality)**.** *Let M have a finite good cover and be oriented of dimension n. Then the map*

$$H^q(M) \to H^{n-q}_c(M)^*$$

defined by

$$[\omega] \mapsto ([\alpha] \mapsto \int_M \omega \wedge \alpha)$$

is an isomorphism.

Note 13.5. The finite cover condition can be removed.

Note 13.6. If M is oriented, compact and connected then integration of n-forms defines an isomorphism $H^n(M) = \mathbb{R}$.

Lemma 13.11 (Five Lemma)**.** *In the commuting diagram below assume that both rows are exact and $\alpha, \beta, \gamma, \epsilon$ are isomorphisms then γ is an isomorphism.*

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow \cdots \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\epsilon} & \\ \cdots \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow \cdots \end{array}$$

Corollary 13.12. *If M is compact and oriented of dimension n then $H^q(M) \simeq H^{n-q}(M)$ for all $q = 0, \ldots, n$.*

13.3. **Kunneth Formula.**

Proposition 13.13 (Kunneth Formula)**.** *If M has a finite good cover then*

$$H^r(M \times N) = \bigoplus_{p+q=r} H^p(M) \otimes H^q(N).$$

for all $r = 0, \ldots, \dim(M \times N)$.

Note 13.7. The map into the Kunneth formula is defined as follows. Let $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ be the projections. If $[\alpha] \otimes [\beta] \in H^p(M) \otimes H^q(N)$ then $\pi^*_M(\alpha) + \pi^*_N(\beta) \in \Omega^{p+q}(M \times N)$ and induces a map $H^p(M) \otimes H^q(N) \to H^{p+q}(M \times N)$.

Note 13.8. Again we can get away without the finite good cover condition.

Appendix A. Partitions of unity.

If M is a manifold a partition of unity is a collection of smooth non-negative functions $\{\rho_\alpha\}_{\alpha\in I}$ such that every $x \in M$ has neighbourhood on which only a finite number of the ρ are non-vanishing and such that $\sum_{\alpha\in I}\rho_\alpha = 1$.

Recall that if $f\colon M \to \mathbb{R}$ is smooth function then we define $\mathrm{supp}(f)$ to be the closure of the set on which f is non-zero. There are two basic existence results on a paracompact, Hausdorff manifold.

(1) If $\{U_\alpha\}_{\alpha\in I}$ is an open cover of M there is a partition of unity $\{\rho_\alpha\}_{\alpha\in I}$ with $\mathrm{supp}(\rho_\alpha) \in U_\alpha$. Such a partition of unity is called subordinate to the cover.
(2) If $\{U_\alpha\}_{\alpha\in I}$ is an open cover of M there is a partition of unity $\{\rho_\alpha\}_{\beta\in J}$, with a possibly different indexing set J such that each $\mathrm{supp}(\rho_\beta)$ is compact and in some U_α.

Appendix B. Vector fields and the tangent bundle.

We have seen how to define tangent vectors at a point of a manifold. In many problems we are interested in vector fields, that is a choice of vector at every point of a manifold. We can think of this in the following manner. Take the union of all the tangent spaces, denote it by

$$TM = \bigcup_{x\in M} T_xM$$

and call it the *tangent bundle* to M. There is an important map $\pi\colon TM \to M$ called the *projection* that sends a vector $X \in T_xM$ to the point $\pi(X) = x$ at which it is located. A vector field is a map $X\colon M \to TM$ with the special property that $X(x) \in T_xM$. This property can be also written as $\pi \circ X = \mathrm{id}_M$, that is $\pi(X(x)) = x$. Such a map $X\colon M \to TM$ is called a *section* of the projection map π. We want to consider smooth vector fields and as we already have a notion of smooth function between manifolds the simplest way to define smooth vector fields is to make TM a manifold. To do this involves a construction that we will use again later so we ill state it in more general form that immediately necessary.

Let E be a set with a surjection $\pi\colon E \to M$ where M is a manifold. Denote by E_x the fibre of E over x, that is the set $\pi^{-1}(x)$. Let V be a finite dimensional vector space. Assume that we can cover M by co-ordinate charts $\{(U_\alpha, \psi_\alpha\}_{\alpha\in I}$ such that for every $\alpha \in I$ and for every $x \in U_\alpha$ there is a bijection

$$\phi_\alpha(x)\colon E_x \to V$$

such that the map

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \to & GL(V) \\ x & \mapsto & \phi_\alpha(x)\circ\phi_\beta(x)^{-1} \end{array}$$

is smooth where $GL(V)$ is the group of all linear isomorphisms of V. Then it is possible to make E a manifold as follows. We define bijections

$$\begin{array}{cccc} \chi_\alpha & : & \pi^{-1}(U) & \to & U \times V \\ & & x & \mapsto & (\pi(x), \phi_\alpha(\pi(x))v) \end{array}$$

To make these into charts we should really identify V with some $\mathbb{R}^k$ but we will not bother to do that. To check compatibility we note that $\chi_\alpha(U_\alpha \cap U_\beta) = U_\alpha \cap U_\beta \times V$ which is open in $\mathbb{R}^n \times V$. Likewise for $\chi_\beta(U_\alpha \cap U_\beta)$. Then the map we want to check is smooth is the map

$$U_\alpha \cap U_\beta \times V \to U_\alpha \cap U_\beta \times V$$

which sends (x, v) to $(x, \phi_\alpha(x)\circ\phi_\beta(x)^{-1}v)$ and this is smooth and invertible. By interchanging α and β we deduce that this map is a diffeomorphism. Hence we have made E into a manifold. Notice that with this manifold structure the map χ_α is a diffeomorphism, as the co-ordinate charts of a manifold are diffeomorphisms. Notice also that each E_x is a vector space from Proposition 4.12. Moreover it easy to check that the addition and scalar multiplication are smooth. Define a section of $\pi\colon E \to M$ to be a smooth map $s\colon M \to E$ which satisfies $s(x) \in E_x$ for all $x \in M$. If s is such a section then on restriction to U_α we can define a map $s_\alpha\colon U \to V$ by $s_\alpha(x) = \phi_\alpha(x)(s(x))$. The s_α are clearly smooth. The converse is also true if s is any map and the s_α defined in this way are smooth then s is smooth.

Consider now the case of the tangent bundle. Let (U_α, ψ_α) be a co-ordinate chart on M. Then $V = R^n$ and $\phi_\alpha(x) = d\psi_\alpha(x)$. The condition we require to hold is that the map

$$x \mapsto d\phi_\beta(\psi^{-1}(x)) \circ d\psi_\alpha^{-1}(x) = d(\psi_\beta \circ \psi_\alpha^{-1})(x) = d_i(\psi_\beta^j \circ \psi_\alpha^{-1})(x)$$

is smooth. But this is just the Jacobian matrix of partial derivatives which depends smoothly on x.

We can now define

Definition B.1. A smooth vector field on a manifold M is a smooth section of the tangent bundle.

To understand what it means to be smooth in terms of co-ordinates recall the definition of $d\psi(x)$. We have the co-ordinate vector fields $(\partial/\partial\psi^i)(x)$ for $i = 1, \ldots, n$. Then

$$d\psi(x)(\frac{\partial}{\partial\psi^i})(x) = e^i$$

where e^i is the standard basis vector of $\mathbb{R}^n$. So clearly this is a smooth map so that the co-ordinate vector fields are smooth.

More generally if X is a vector field we can write it as

$$X(x) = \sum_{i=1}^n X^i(x)\frac{\partial}{\partial\psi^i}(x).$$

for any $x \in U$, and functions $X^i\colon U \to \mathbb{R}$. Then

$$d\psi(x)(X(x)) = (X^1(x), \ldots, X^n(x)).$$

This proves

Proposition B.2. *Let X be a vector field on a manifold M. Then if X is smooth and (U, ψ) is a co-ordinate chart then if we let*

$$X(x) = \sum_{i=1}^n X^i(x)\frac{\partial}{\partial\psi^i}(x)$$

the functions $X^i\colon U \to \mathbb{R}$ are smooth. Conversely if X is a vector field and we can cover M with co-ordinate charts (U, ψ) such that the corresponding $X^i\colon U \to \mathbb{R}$ are smooth then X is smooth.

Appendix C. Vector fields and derivations.

Let us now define derivations of $C^\infty(M)$.

Definition C.1. A derivation of $C^\infty(M)$ is a linear map

$$D\colon C^\infty(M) \to C^\infty(M)$$

such that

$$D(fg) = D(f)g + fD(g).$$

A vector field X gives rise to a derivation $f \mapsto X(f)$ and using the previous Lemma we have

Proposition C.2. *Every derivation arises from a vector field.*

Proof. Let D be a derivation. Then note that for any x $f \mapsto D(f)(x)$ is a derivation at x. Hence there is a tangent vector $X(x)$ such that $D(f)(x) = X(x)(f)$ for all x. We have to check that $X(x)$ depends smoothly on x. But if we choose local co-ordinates ψ as in the proof above and extend them to global functions ψ then we have

$$X(x) = \sum_{i=1}^n D(\psi^i)(x)\frac{\partial}{\partial\psi^i}(x)$$

but $D(\psi)$ is a smooth function so by Proposition B.2 $X(x)$ is smooth. ∎

The advantage of thinking of a vector field as a derivation is that derivations have a natural bracket operation. If D and D' are two derivations then a simple calculation shows that $[D, D']$ defined by

$$[D, D'](f) = D(D'(f)) - D'(D(f)).$$

is also a derivation. So we can define the bracket of two vector fields X and Y and called the Lie bracket $[X, Y]$. To calculate $[X, Y]$ we apply it to ψ^i then we have

$$[X, Y](\psi^i) = X(Y(\psi^i)) - Y(X(\psi^i))$$

so that if

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial \psi^i}$$

and

$$Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial \psi^i}$$

so that

$$[X, Y](\psi^i) = \sum_{j=1}^{n} X^j \frac{\partial Y^i}{\partial \psi^j} - Y^j \frac{\partial X^i}{\partial \psi^j}.$$

Hence

$$[X, Y] = \sum_{i,j=1}^{n} (X^j \frac{\partial Y^i}{\partial \psi^j} - Y^j \frac{\partial X^i}{\partial \psi^j}) \frac{\partial}{\partial \psi^i}.$$

Appendix D. Tensor products

If V and W are finite dimensional vector spaces then the Cartesian product $V \times W$ is naturally a vector space called the direct sum of V and W and denoted $V \oplus W$. The tensor product is a more complicated object. To define it we start by defining for any *set* X the free vector space over X, $F(X)$. This is the set of all maps from X to $\mathbb{R}$ which are zero except at a finite number of points. We define the vector space structure by adding and scalar multiplying maps. Each x gives rise to a function $\delta(x)$ which is one at x and zero elsewhere. We therefore have a map $\delta \colon X \to F(X)$. By construction the span of the image of δ is all of $F(X)$.

The special property of the free vector space over X is the following.

Proposition D.1. *Let $f \colon X \to U$ be any map from X into a vector space U then there is a unique linear map $\hat{f} F(X) \to U$ such that $\hat{f} \circ \delta = f$.*

Proof. The general element of $F(X)$ is

$$\sum_{i=1}^{n} a_i \delta(x_i)$$

for $a_i \in \mathbb{R}$. We define

$$\hat{f}(\sum_{i=1}^{n} a_i \delta(x_i)) = \sum_{i=1}^{n} a_i f(x^i).$$

□

Given two vector spaces V and W we can define $F(V \times W)$. This is an infinite dimensional vector space. We shall denote $\delta((v, w))$ by $\delta(v, w)$. Consider the subspace Z defined as the span of all elements of the form

$$\delta(\lambda v + \mu v', w) - \lambda \delta(v, w) - \mu \delta(v', w)$$

and

$$\delta(v, \lambda w + \mu w') - \lambda \delta(v, w) - \mu \delta(v, w')$$

for any real numbers λ and μ and vectors $v, v' \in V$ and $w, w' \in W$. Let us denote

$$V \otimes W = F(V \times W)/Z$$

and define a map

$$\otimes\colon V \times W \to V \otimes W$$

by

$$v \otimes w = \delta(v, w) + Z.$$

We have

Proposition D.2. *The map* $\otimes\colon V \times W \to V \otimes W$ *is bilinear.*

Proof. We check the first factor only

$$\begin{aligned}(\lambda v + \mu v') \otimes w =& \delta(\lambda v + \mu v', w) + Z \\ =& \delta(\lambda v + \mu v', w) - \lambda\delta(v, w) \\ & - \mu\delta(v', w) + \lambda\delta(v, w) + \mu\delta(v', w) + Z \\ =& \lambda\delta(v, w) + \mu\delta(v', w) + Z \\ =& \lambda(\delta(v, w) + Z) + \mu(\delta(v', w) + Z) \\ =& \lambda v \otimes w + \mu v' \otimes w\end{aligned}$$

∎

From Proposition D.1 we know that any map $f\colon V \times W \to U$, where U is a vector space extends to a map $\hat{f}\colon F(V \times W) \to U$. Standard linear algebra tells us that we can take the quotient to get a map $\tilde{f}\colon V \otimes W \to U$ if $\hat{f}(Z) = 0$. The map is defined by $v \otimes w \to f(v, w)$. For example if $v^* \in V^*$ and $w^* \in W^*$ then $v \otimes w \to v^*(v)w^*(w)$ defines a linear map from $V \otimes W \to \mathbb{R}$.

Let $\{v^1, \ldots, v^n\}$ be a basis of V and $\{w^1, \ldots, w^m\}$ be a basis of W. Consider the set of mn vectors $v^i \otimes w^j$ in $V \otimes W$. We wish to show that they form a basis. First we check that they span the space $V \otimes W$. As the elements of $V \otimes W$ are finite linear combinations of elements of the form $v \otimes w$ it suffices to show that these are all in the span of the vectors $v^i \otimes w^j$. But this follows from the bilinearity. If $v = \sum_{i=1}^n a_i v^i$ and $w = \sum_{j=1}^m b_j w^j$ then

$$v \otimes w = \sum_{i=1}^n \sum_{j=1}^m a_i b_j v^i \otimes w^j.$$

To show that they are linearly independent assume that

$$0 = \sum_{i=1}^n \sum_{j=1}^m a_{ij} v^i \otimes w^j.$$

Let v_i^* and w_j^* be the dual bases of $V^*$and W^*. That is $v_i^*(v^j) = \delta_i^j$ and $w_i^*(w^j) = \delta_i^j$. Then apply the map $V \otimes W \to \mathbb{R}$ defined by v_i^* and w_j^* to this equation to obtain $a_{ij} = 0$. So we have proved.

Proposition D.3. *If V and W are finite dimensional vector spaces then*

$$\dim(V \otimes W) = \dim(V)\dim(W).$$

We can iterate tensor products. If V and W and U are vector spaces we can form $(V \otimes W) \otimes U$ and $V \otimes (W \otimes U)$. These different vector spaces are in fact isomorphic via the map

$$(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u).$$

We use this map to identify these two spaces and ignore the brackets. We write $V \otimes U \otimes W$ for the triple tensor product. More generally we can form finitely many tensor products.

We also need to know about tensor products of maps. If $X\colon V \to V'$ is linear and $Y\colon W \to W'$ is linear then we can define a map

$$V \times W \to V' \otimes W'$$

by $(v, w) \mapsto X(v) \otimes Y(v)$. This is a bilinear map so factors to a map $V \otimes W \to V' \otimes W'$ which we denote by $X \otimes Y$. It is defined by $(X \otimes Y)(v \otimes w) = X(v) \otimes Y(w)$.

We have seen that any bilinear map $V \times W \to \mathbb{R}$ gives rise to a linear map $V \otimes W \to \mathbb{R}$. It is easy to show that this is an isomorphism. More generally if for any collection of vector spaces $V_1, \dots, V_k$ we denote by $\mathrm{Mult}(V_1 \times \cdots \times V_k, \mathbb{R})$ the space of all multilinear maps from $V_1 \times \dots V_k \to \mathbb{R}$ we have

Proposition D.4. *If $V_1, \dots, V_k$ are vectors spaces then there is a natural isomorphism*

$$Mult(V_1 \times \cdots \times V_k, \mathbb{R}) \to (V_1 \otimes V_2 \otimes \cdots \otimes V_k)^*$$

defined by

$$f \mapsto (v_1 \otimes \cdots \otimes v_k \mapsto f(v_1, \dots, v_k)).$$

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