Differential Geometry. Exercises.

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Contents

1 Co-ordinate charts and manifolds.

Exercise 1.1. Consider the n sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}.$$

Let

$$U_i = \{x \in S^n \mid x^i \neq 1\} = S^n - \{e_i\}$$

where e_i has all components 0 except the ith which is equal to one. If x is a point in U_i show that there is a unique line through x and the vector e_i. Show that this line intersects the plane

$$\{x \mid x^i = 0\}$$

in exactly one point. Writing this point as

$$(\psi_i^1(x), \psi_i^2(x), \dots, \psi_i^{i-1}(x), 0, \psi_i^i(x), \dots, \psi_i^n(x))$$

defines a function

$$\psi_i = (\psi_i^1, \dots, \psi_i^n)\colon U_i \to \mathbb{R}^n.$$

Show that (U_i, ψ_i) is a co-ordinate chart on S^n and that

$$\{(U_i, \psi_i) \mid i = 1, \dots, n\}.$$

is an atlas for S^n.

The functions ψ_i are are said to arise by *stereographic projection* from e_i onto the plane $\{x \mid x^i = 0\}$.

Exercise 1.2. Consider the sphere S^n again. Define

$$U_i^+ = \{x \in S^n \mid x^i > 0\}$$

and define $\psi_i^+ : U_i^+ \to \mathbb{R}^n$ by

$$\psi_i^+(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n).$$

Show that (U_i^+, ψ_i^+) is a co-ordinate chart for S^n. Similarly define

$$U_i^- = \{x \in S^n \mid x^i < 0\}$$

and define $\psi_i^- : U_i^+ \to \mathbb{R}^n$ by

$$\psi_i^-(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n).$$

Again show that (U_i^-, ψ_i^-) is a co-ordinate chart for S^n.

Show that

$$\{(U_i^+, \psi_i^+), (U_i^-, \psi_i^-) \mid i = 1, \dots, n\}$$

is an atlas for S^n.

Exercise 1.3. Show that the atlases in Exercises 1.1 and 1.2 define the same maximal atlas on S^n.

Exercise 1.4. Let $\mathbb{R}P_n$ be the set of all lines (through the origin) in $\mathbb{R}^{n+1}$. This space is called real, projective space of dimension n. If x is a non-zero vector in $\mathbb{R}^{n+1}$ denote by $[x]$ the line through x. Show that $[x] = [y]$ if and only if there is a non-zero real number λ such that $x = \lambda y$.

Define subsets U_i of $\mathbb{R}P_n$ by

$$U_i = \{[x^0, \dots, x^n] | x^i \neq 0\}$$

and maps $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x^0, \dots, x^n]) = (\frac{x^0}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^n}{x_i})$$

for every $i = 0, \dots, n$. Show that φ_i is well defined and that (U_i, φ_i) is a co-ordinate chart on $\mathbb{R}P_n$. Show that

$$\{(U_i, \varphi_i) | i = 0, \dots, n\}$$

is an atlas for $\mathbb{R}P_n$.

Exercise 1.5. Show that if M_1 and M_2 are manifolds then there is a natural way of making $M_1 \times M_2$ into a manifold so that $\dim(M_1 \times M_2) = \dim(M_1) + \dim(M_2)$.

Exercise 1.6. Repeat exercise (1.4) for $\mathbb{C}^n$ to define $2n$ dimensional complex projective space $\mathbb{C}P_n$ as the space of complex lines through zero in $\mathbb{C}^{n+1}$.

2 Smooth functions.

Exercise 2.1. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \exp\left(\frac{-1}{1-x^2}\right) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

and show that h is smooth. By integrating h find a smooth function $g\colon \mathbb{R} \to \mathbb{R}$ with the property that $g(x)$ is zero for $x < -1$ and $g(x)$ is one for $x > 1$. Show that for any $\epsilon > \delta > 0$ there is a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ with $\phi(x)$ equal to zero if $\|x\| > \epsilon$ and ϕ equal to one if $\|x\| < \delta$. Now consider a manifold M and a point x. By using co-ordinates show that if U is any open subset of M containing x then there are open subsets U_1 and U_2 with $x \in U_1 \subset U_2 \subset U$ and a smooth function $f\colon M \to \mathbb{R}$ with f equal to 1 on all of U_1 and equal to zero outside of U_2.

Exercise 2.2. Let x be point in a manifold M. Let X_x be the set of all pairs (U, f) where U is a open set containing x and $f\colon U \to \mathbb{R}$ is a smooth function. Define a relation on X_x by saying that $(U, f) \simeq (V, g)$ if there is an open set W with $x \in W \subset U \cap V$ and $f_{|W} = g_{|V}$. Show that this an equivalence relation. Equivalence classes are called *germs* at x and the set of them we will denote by G_x. Show that G_x is an algebra under pointwise addition, scalar multiplication and multiplication. If $f \in C^\infty(M, \mathbb{R})$ the algebra of all smooth functions on M it defines the germ containing (M, f). Show that the map this induces $C^\infty(M, \mathbb{R}) \to G_x$ is onto. [Hint: Use 2.1.]

Exercise 2.3. Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$F(x, y, z) = (x^2 + y^2 + z^2 - 9, x + y + z - 3).$$

If we identity the tangent spaces to $\mathbb{R}^3$ and $\mathbb{R}^2$ with $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively calculate the tangent map

$$T_{(x,y,z)}F\colon \mathbb{R}^3 \to \mathbb{R}^2.$$

Exercise 2.4. Define a map $F\colon S^2 \to \mathbb{C}P_1$ by

$$F(x, y, z) = [x + iy, 1 - z].$$

By using the co-ordinates defined in Exercises (1.1) and (1.6) show that this map is well defined as $z \to 1$ and that it is, in fact, a diffeomorphism.

3 Submanifolds.

Exercise 3.1. Show that the set defined by the equation

$$r^2 - a^2 = (\sqrt{x^2 + a^2} - a)^{1/2}$$

is a smooth submanifold of $\mathbb{R}^3$ if a and r are real numbers with $r < a$.

Exercise 3.2. Show that the following subset of $\mathbb{R}^3$ is a submanifold:

$$Q = \{(x, y, z) \mid x^2 + y^2 + z^2 = 9 \quad \text{and} \quad x + y + z = 3\}.$$

4 Vector fields and differential forms.

Exercise 4.1. Let X and Y be vector fields on a manifold M. Define a new vector field $[X,Y]$ by defining it in local co-ordinates (U,ϕ) by

$$[X,Y]_{|U} = \sum_{i,j} \big(X_i \frac{\partial}{Y}^{j} \partial\phi^i - Y_i \frac{\partial X^j}{\partial \phi^i} \big) \frac{\partial}{\partial \phi^j}.$$

Show that this makes sense. That is it doesn't really depend on the choice of co-ordinates. The vector field $[X,Y]$ is called the Lie bracket of X and Y.

Exercise 4.2. If X is a vector field and ω is a differential 1-form show that the differential 1-form defined by

$$L_X(\omega) = \sum_{i,j} \big(X^i \frac{\partial \omega_j}{\partial \theta^i} + \omega_i \frac{\partial X^i}{\partial \theta^j} \big) d\theta^j.$$

where

$$X = \sum_i X^i \frac{\partial}{\partial \theta^i} \quad \text{and} \quad \omega = \sum_i \omega_i d\theta^i$$

is actually independent of the choices of co-ordinates. We call $L_X(\omega)$ the Lie derivative of ω by X.

Exercise 4.3. Let α and β be p and q forms, respectively on a manifold M. Show that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Exercise 4.4. Consider the circle $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$. This is a manifold of dimension 1. The circle has a co-ordinate chart (U,θ) where $U = S^1 - \{(1,0)\}$ and $\theta \colon U \to (0, 2\pi)$ is defined implicitly by

$$(x,y) = (\cos(\theta(x,y)), \sin(\theta(x,y))).$$

That is θ is the usual angle co-ordinate in polar co-ordinates. Identify the tangent space to the circle at (x,y) with the line in $\mathbb{R}^2$ tangential to the circle at (x,y). Calculate a formula for the vector field $\partial/\partial\theta$ in terms of x and y and hence show that it extends from U to a vector field on all of S^1. Show that $d\theta$ also extends to a differential 1-form ω on all of the circle. Show that there is no function $f : S^1 \to \mathbb{R}$ such that $\omega = df$.

Exercise 4.5. Let $S^2 = \{x \in \mathbb{R}^3 | \|x\|^2 = 1\}$ be the two-sphere. Recall that the spherical co-ordinates (θ, ϕ) of the point (x,y,z) on the two-sphere are defined by requiring that:

$$\begin{aligned} x &= \sin(\psi)\cos(\theta) \\ y &= \sin(\psi)\sin(\theta) \\ z &= \cos(\psi). \end{aligned}$$

Find an open set $U \subset S^2$ for the domain of the spherical co-ordinates so that $\psi \in (0,\pi)$ and $\theta \in (0, 2\pi)$.

For any x in S^2 and $X, Y \in T_x S^2$ define a differential two-form ω on S^2 by $\omega_x(X,Y) = \langle x, X \times Y \rangle$, where $\langle\ ,\ \rangle$ is the usual inner-product on $\mathbb{R}^3$ and x is the cross-product of three vectors. By using suitable co-ordinates (spherical are good) calculate the integral of ω over S^2 and show that it is non-zero.

Exercise 4.6. Show that it is not possible to find a differential one-form μ on the two sphere such that $d\mu$ is the volume form ω defined in exercise (4.5).

Exercise 4.7. Consider the torus T^2 in $\mathbb{R}^3$ with co-ordinates (θ, ϕ) defined implicitly by

$$x = (b + a\sin(\phi))\cos(\theta), (b + a\sin(\phi))\sin(\theta), a\cos(\phi)).$$

Calculate $\partial/\partial\psi$ and $\partial/\partial\theta$. Calculate the (outward) unit normal $n(x)$ to the torus, this is the vector in $\mathbb{R}^3$ orthogonal to the tangent space to the torus at x. You will need to draw a picture or something to check it is the outward normal.

Define vol a two-form by $\mathrm{vol}(X, Y) = \langle n, X \times Y\rangle$ and calculate its integral over T^2 when we orient T^2 in such a way as to make vol positive.

Exercise 4.8. Recall the definition of $\mathbb{R}P_2$ the space all lines through the origin in $\mathbb{R}^2$ and its associated co-ordinate charts given in Exercise 1.4. Calculate the linear relationship between the basis of one forms $d\psi_i^1, d\psi_i^2$ and the basis of one forms $d\psi_j^1, d\psi_j^2$ for $i \neq j$. Hence calculate the relationship between $d\psi_i^1 \wedge d\psi_i^2$ and $d\psi_j^1 \wedge d\psi_j^2$. Show that $\mathbb{R}P_2$ is not orientable.

Exercise 4.9. Let $f\colon M \to N$ be a smooth map. If ω is a p-form on N show that $df^*(\omega) = f^*d\omega$.

5 Complex line bundles

Exercise 5.1. Let ∇^0 and ∇^1 be connections on a complex line bundle L and define

$$\nabla^t(\phi) = t\nabla^1(\phi) + (1-t)\nabla^0(\phi)$$

for any section ϕ of L. Show that ∇^t is a connection for any real number t. Calculate its curvature.

Exercise 5.2. Show that if $L \to M$ is a trivial bundle then it has zero Chern class.

Exercise 5.3. Consider the Hopf bundle H over $\mathbb{C}P_1$. Define parameters on $U_0 = \mathbb{C}P_1 - [1, 0]$ by $(x, y) \mapsto [x+iy, 1]$. Let $s_0([x+iy, 1]) = ([x+iy, 1], (x+iy, 1))$ be the section defined in class. Using (hermitian) orthogonal projection define a connection ∇ on H and calculate the connection one form A_0. Be careful to make the orthogonal projection complex linear. Calculate the curvature over the open set U_0 and integrate it over U_0 to find the Chern class of H. You may find it convenient to work with the complex differential forms $dz = dx + idy$ and $d\bar{z} = dx - idy$.

Exercise 5.4. Consider the tangent bundle to the two-sphere. Give it the connection defined by orthogonal projection and calculate its curvature and hence the chern class of the tangent bundle to the two-sphere.

Exercise 5.5. Repeat Exercise 5.4 for the torus using the co-ordinates defined in Exercise 4.7.

Exercise 5.6. This assumes you are familiar with the Gauss-Bonnet theorem. If Σ is a closed surface in $\mathbb{R}^3$ define a connection on its tangent bundle by using orthogonal projection. Relate the curvature of this connection to the usual Gaussian curvature.