# Real Bundle Gerbes

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19th May 2017

# Introduction

- Joint work with Pedram Hekmati (Auckland), Richard Szabo (Heriot-Watt) and Raymond Vozzo (Adelaide)
- arXiv:1608.06466
- *Real bundle gerbes orientifolds and twisted KR-homology*

## Outline

- Real spaces
- Real and $\mathbb{Z}_2$-equivariant bundles
- Grothendieck cohomology
- Bundle gerbes
- Real bundle gerbes
- Jandl gerbes
- Final example

# Real spaces

**Definition**

A *Real space* is a pair $(M, \tau)$ where $M$ is a manifold and $\tau \colon M \to M$ is a smooth involution ($\tau^2 = 1$).

Call $\tau$ a *Real structure* on $M$. Note that a Real structure is the same thing as a $\mathbb{Z}_2$ action on $M$.

Some of you will recognise this definition from Atiyah's work on Real $K$-theory [1]. Although he didn't use the upper case $R$.

We are interested in lifting the Real structure on $M$ to various kinds of geometric objects living over $M$. We illustrate the idea with an example.

## Example

Let $f\colon M \to U(1)$. Define $\tau(f)\colon M \to U(1)$ by $\tau(f)(m) = \overline{f(\tau(m))}$. (I will abuse notation by calling just about every involution $\tau$.) Notice that $\tau^2(f) = f$. Call $f$ *Real* if $\tau(f) = f$ or $f(x) = \overline{f(\tau(x))}$. Two examples are worth bearing in mind.

1. Let $\tau = \mathrm{id}_M\colon M \to M$. Then $f$ is Real precisely if $f(x) = \overline{f(x)}$, that is $f$ is a real ($\mathbb{Z}_2$) valued function.
2. Let $M = X \times \{0, 1\}$ and define $\tau(x, 0) = (x, 1)$ and $\tau(x, 1) = (x, 0)$. Then $f\colon M \to U(1)$ is a pair of functions $f_0, f_1\colon X \to U(1)$ where $f(x, i) = f_i(x)$. Then $f$ is Real if and only if $f_0 = \bar{f}_1$. So the theory of Real functions on $M$ is the theory of complex ($U(1)$) functions on $X$.

So Real functions generalise both real valued functions and ordinary complex functions. Or in Atiyah's work Real $K$-theory generalises $KO$ and ordinary $K$.

We say that two Real structures $\tau$ and $\tau'$ are *equivalent* if there is a diffeomorphism $\rho\colon M \to M$ with $\tau' = \rho^{-1}\tau\rho$.

**Example (Inequivalent Real structures on $S^2$ [2])**

On the two-sphere $S^2$ it turns out everything is equivalent to an orthogonal transformation which must have eigenvalues $\pm 1$. So there are $4$ inequivalent structures: (a) the identity map (b) reflection in the $x - y$ plane $(x, y, z) \mapsto (x, y, -z)$ (c) rotation by $\pi$ around $z$-axis $(x, y, z, ) \mapsto (-x, -y, z)$ and (d) the antipodal map $(x, y, z) \mapsto (-x, -y, -z)$.

**Example ([7] How many inequivalent Real structures on $S^1 \times S^1$ or $S^1 \times S^2$?)**

The space $S^1 \times S^1$ has $5$ classes of Real structures. The space $S^1 \times S^2$ has $13$ classes of Real structures.

# What is a $U(1)$ principal bundle ?

Let $L \to M$ be a complex line bundle. Assume further that each $L_m$ has a Hermitian inner product. Let $P \subset L$ be the set of all unit vectors. Then if $v \in P$ then $zp \in P$ if and only if $z\bar{z} = 1$ that is $z \in U(1)$. So $P$ is acted on by $U(1)$. In fact each fibre $P_m$ of $P \to M$ is a circle rotated by $U(1)$ as it is set of unit vectors in the one-dimensional complex vector space $L_m$.

If $f\colon M \to \mathbb{C}$ and we define a function $L \to L$ by $L_m \ni v \mapsto f(m)v$ then this function preserves $P$ if and only if $f\colon M \to U(1)$.

In general $P \to M$ is a $U(1)$ bundle if each $P_m$ looks like a circle which is rotated by $U(1)$ and there is some appropriate local triviality condition. They all arise from Hermitian line bundles as above.

# Real and $\mathbb{Z}_2$-(equivariant) bundles

**Definition**

Let $(M, \tau)$ be a Real space. We say that a $U(1)$-bundle, $P \to M$ is a *Real bundle* if there is an conjugate involution $\tau \colon P \to P$, i.e. $\tau(pz) = \tau(p)\bar{z}$ and $\tau^2 = 1$.

Notice the important fact that the Real structure $\tau \colon P \to P$ is an isomorphism $\tau^{-1}(P) \simeq P^*$. This gives a necessary condition for the existence of a Real structure. But we shall see it's not sufficient because we also need $\tau^2 = 1$.

**Definition**

Let $(M, \tau)$ be a Real space. We say that a $U(1)$-bundle, $P \to M$ is a $\mathbb{Z}_2$ *(equivariant) bundle* if there is an involution $\tau \colon P \to P$, i.e. $\tau(pz) = \tau(p)z$ and $\tau^2 = 1$. That is the $\mathbb{Z}_2$ action lifts to $P$.

Similarly for this to exist we must have $\tau^{-1}(P) \simeq P$ but that isn't sufficient.

Consider the problem of classifying Real structures on $P$ up to the obvious equivalences obtained from bundle automorphisms.

Every bundle automorphism $P \to P$ is multiplication by a map $f : M \to U(1)$. If $\tau$ is a Real structure on $P$ we can get another Real structure by forming $\hat{\tau} = f\tau f^{-1} = f\tau\bar{f}$. That is if $p \in P_m$ then $\hat{\tau}(p) = f(\tau(m))\tau(\bar{f}(m)p)$ or $\hat{\tau} = (f \circ \tau)\tau\bar{f} = (f \circ \tau)f\tau$. In such a case we say that $\tau$ and $\hat{\tau}$ are *equivalent*.

We must also have that $\hat{\tau}$ satisfies $\hat{\tau} = g\tau$ for $g : M \to U(1)$. We require $\hat{\tau}^2 = 1$ or $g(\bar{g} \circ \tau) = 1$.

So $\tau$ and $\hat{\tau}$ are equivalent if $(\bar{f} \circ \tau)g\tau = f\tau$ or $g = f(f \circ \tau)$. It turns out that if $M$ is simply-connected we can always solve this equation.

**Proposition**

*If $M$ is simply connected any bundle $P$ has at most one Real structure up to equivalence.*

On simple manifolds simple calculations and arguments let you determine the possible Real bundles.

# Real bundles on $S^2$

## Example

Consider the Real structures on $S^2$ discussed before. Recall that $c(\deg(\tau^{-1}(P))) = \deg(\tau)c(P)$ and $\deg(\tau) = \pm 1$ because $\tau^2 = 1$.
(a) (b) $\tau$ homotopic to $\mathrm{id}_{S^2}$. $\deg(\tau) = +1$. Hence $\tau^{-1}(P) \simeq P$. But to admit a Real structure we must have $\tau^{-1}(P) \simeq P^*$ which is only possible if $c(P) = 0$ or $P$ is trivial.
(c) $\tau$ is the antipodal map. $\deg(\tau) = -1$ Identify $S^2 = \mathbb{C}P_1$. Then $\tau([z_0, z_1]) = [\bar{z}_1, -\bar{z}_0]$. Consider the Hopf bundle $H \to \mathbb{C}P_1$ whose fibre at $[z_0, z_1]$ is all $(w_0, w_1)$ of unit length and in the line $[z_0, z_1]$. Then we can lift $\tau$ to an antilinear $\tau \colon H \to H$ by $\tau(w_0, w_1) = (\bar{w}_1, -\bar{w}_0)$. But $\tau^2 = -1$. Can we fix this by replacing $\tau$ by $g\tau$? It turns out that we can't. However notice that any even power of $H$ admits a Real structure.
(d) $\tau$ is the reflection. $\deg(\tau) = -1$. On $\mathbb{C}P_1$ this is conjugation $\tau([z_0, z_1]) = [\bar{z}_0, \bar{z}_1]$. Now $\tau$ lifts to the Hopf bundle and squares to $1$ so every bundle admits a Real structure.

So the necessary topological conditions for existence of Real structures are not sufficient.

Notice that if we call a $\tau$ with $\tau^2 = -1$ *Quaternionic* then the odd powers of the Hopf bundle have unique Quaternionic structures for the antipodal map (c). This turns out to be typical.

**Proposition**

*If $M$ is simply connected and Real then a $U(1)$-bundle $P \to M$ either has $\tau^{-1}(P) \ncong P^*$ or it has a unique Real or Quaternionic structure (up to equivalence).*

To get further in this we need a cohomological classification of Real and $\mathbb{Z}_2$-equivariant bundles.

# Grothendieck cohomology

In his famous Tohoku paper, Grothendieck introduced a cohomology theory for sheaves with group actions. We will be concerned with the case that the group is the cyclic group $\mathbb{Z}_2$.

We use the notation $\mathcal{U}(1)$ for the sheaf of functions with values in $U(1)$, also the same sheaf where $\mathbb{Z}_2$ acts trivially and $\overline{\mathcal{U}(1)})$ for the same sheaf with $\mathbb{Z}_2$ acting such that non-trivial elements acts by conjugation.

Some of the groups of interest to us are:

- $H^0(M;\mathbb{Z}_2,\overline{\mathcal{U}(1)})$: functions $f : M \to U(1)$ satisfying $\bar{f} \circ \tau = f$
- $H^0(M;\mathbb{Z}_2,\mathcal{U}(1))$: functions $f : M \to U(1)$ satisfying $f \circ \tau = f$
- $H^1(M;\mathbb{Z}_2,\overline{\mathcal{U}(1)})$: $U(1)$ bundles $P \to M$ with a Real structure
- $H^1(M;\mathbb{Z}_2,\mathcal{U}(1))$: $U(1)$ bundles $P \to M$ with a $\mathbb{Z}_2$ structure

Recalling that $H^0(M,\mathcal{U}(1))$ is functions from $M$ to $U(1)$ and $H^1(M,\mathcal{U}(1))$ is $U(1)$-bundles it's possible to fit these together in a useful long exact sequence.

# Useful long exact sequence

$$0 \longrightarrow H^0(M,\mathbb{Z}_2,\overline{\mathcal{U}(1)}) \longrightarrow H^0(M,\mathcal{U}(1)) \xrightarrow{g\mapsto g(g\circ\tau)} H^0(M;\mathbb{Z}_2,\mathcal{U}(1)) \longrightarrow$$

$$\longrightarrow H^1(M;\mathbb{Z}_2,\overline{\mathcal{U}(1)}) \longrightarrow H^1(M,\mathcal{U}(1)) \xrightarrow{P\mapsto P\otimes\tau^{-1}(P)} H^1(M;\mathbb{Z}_2,\mathcal{U}(1)) \longrightarrow$$

$$\longrightarrow H^2(M;\mathbb{Z}_2,\overline{\mathcal{U}(1)}) \longrightarrow H^2(M,\mathcal{U}(1)) \longrightarrow H^2(M;\mathbb{Z}_2,\mathcal{U}(1)) \cdots$$

# What is the geometry ?

$$0 \longrightarrow \text{Real functions} \longrightarrow \text{functions} \xrightarrow{g \mapsto g(g \circ \tau)} \{h \mid h = h \circ \tau\}$$

$$\longrightarrow \text{Real bundles} \xrightarrow{\text{forgetful}} \text{bundles} \xrightarrow{P \mapsto P \otimes \tau^{-1}(P)} \mathbb{Z}_2 \text{ bundles}$$

$$\longrightarrow H^2(M; \mathbb{Z}_2, \overline{\mathcal{U}(1)}) \longrightarrow \text{bundle gerbes} \longrightarrow H^2(M; \mathbb{Z}_2, \mathcal{U}(1)) \cdots$$

The forgetful map is forgetting the Real structure on the bundle.

# Bundle gerbes

Recall that $U(1)$-bundles $P \to M$ satisfy the following properties:

1. Two bundles $P$ and $Q$ can be multiplied to form $P \otimes Q$
2. A bundle $P$ has a dual $P^*$
3. If $f: N \to M$ there is pullback $f^{-1}(P) \to N$.
4. A bundle is determined up to isomorphism by a class $c(P) \in H^1(M, \mathcal{U}(1)) = H^2(M, \mathbb{Z})$.
5. $c(P \otimes Q) = c(P) + c(Q)$, $c(P^*) = -c(P)$ and $c(f^{-1}(P)) = f^*(c(P))$.
6. A bundle $P$ is *trivial* ($P \cong M \times U(1)$) if and only if $c(P) = 0$.
7. There are choices of trivialisation. Any two differ by a function $f: M \to U(1)$.

Bundle gerbes are a geometric object with many of the properties of $U(1)$-bundles with some modifications.

1. they have a characteristic class in $H^3(M, \mathbb{Z})$
2. trivialisation is a little more complicated

We recall the precise definition of a bundle gerbe next. But first we need some notation. If $\pi: Y \to M$ is a surjective submersion we denote by $Y^{[p]}$ the $p$-fold fibre product. That is all $(y_1, \ldots, y_p) \subset Y^p$ with $\pi(y_1) = \cdots = \pi(y_p)$.

# Bundle gerbes

**Definition (M [4])**

A *bundle gerbe* over $M$ is a pair $\mathcal{G} = (P, Y)$ where $Y \to M$ is a surjective submersion and $P \to Y^{[2]}$ is a $U(1)$ bundle satisfying:

- There is a *bundle gerbe multiplication* which is a smooth isomorphism

$$m \colon P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)}$$

  for all $(y_1, y_2, y_3) \in Y^{[3]}$. Here $P_{(y_1,y_2)}$ denotes the fibre of $P$ over $(y_1, y_2) \in Y^{[2]}$.
- This multiplication is associative, that is the following diagram commutes for all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$:

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \to & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \downarrow & & \downarrow \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} & \to & P_{(y_1,y_4)} \end{array}$$

# Basic properties of bundle gerbes

It is straightforward to define products, duals and pullbacks of bundle gerbes. For example if $\mathcal{G} = (P, Y)$ and $\mathcal{H} = (Q, X)$ are bundle gerbes then $Y \times_M X$ is a surjective submersion and we can define $P \otimes Q$ over $(Y \times_M X)^{[2]}$ by

$$(P \otimes Q)_{(y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}.$$

This defines $\mathcal{G} \otimes \mathcal{H} = (P \otimes Q, Y \times_M X)$. Similarly $\mathcal{G}^* = (P^*, Y)$. Pullback is straightforward once you know that surjective submersions pullback.

Slightly more complicated is the definition of the *Dixmier-Douady class* of a bundle gerbe $\mathcal{G} = (P, Y)$ which is a class

$$\mathrm{DD}(\mathcal{G}) \in H^2(M, \mathcal{U}(1)) = H^3(M, \mathbb{Z}).$$

It satisfies $DD(\mathcal{G} \otimes \mathcal{H}) = \mathrm{DD}(\mathcal{G}) + \mathrm{DD}(\mathcal{H})$, $\mathrm{DD}(\mathcal{G}^*) = -\mathrm{DD}(\mathcal{G})$ and $DD(f^{-1}(\mathcal{G})) = f^*(\mathrm{DD}(\mathcal{G}))$.

# Trivialisation

Recall that if $Y \to M$ is a surjective submersion and $R \to Y$ is a $U(1)$-bundle we define a $U(1)$-bundle $\delta(R) \to Y^{[2]}$ by $\delta(Y)_{(y_1,y_2)} = R_{y_2} \otimes R^*_{y_1}$. Contraction defines a natural bundle gerbe multiplication on $\delta(R)$. We have

**Definition**

We call a bundle gerbe $\mathcal{G} = (P, Y)$ *trivial* if it is isomorphic to some $(\delta(R), Y)$ for $R \to Y$ and we call $R \to Y$ a *trivialisation* of $P$.

**Proposition**

*A bundle gerbe has zero Dixmier-Douady class if and only it is trivial.*

Recall that if $P \to M$ and $Q \to M$ are $U(1)$-bundle then an isomorphism from $P$ to $Q$ is a section (trivialisation) of $P^* \otimes Q$. Motivated by this we have

# Stable isomorphism

**Definition**

Two bundle gerbes $\mathcal{G}$ and $\mathcal{H}$ are said to be *stably isomorphic* if $\mathcal{G}^* \otimes \mathcal{H}$ is trivial and a choice of such a trivialisation is called a *stable isomorphism*.

**Proposition**

*Stable isomorphism is an equivalence relation. The stable isomorphism classes of bundle gerbes are classified by* $H^2(M, \mathcal{U}(1)) = H^3(M, \mathbb{Z})$.

# Two asides

## $p$-gerbes

There is a clear parallel between the properties of $U(1)$ bundles and the properties of bundles gerbes with a shift in the dimension of the relevant cohomology group. They form part of a heirarchy of $p$-gerbes whose characteristic class is in $H^{p+2}(M, \mathbb{Z})$ and which include

| | |
|---|---|
| $p = -1$ | functions into $U(1)$ |
| $p = 0$ | $U(1)$-bundles |
| $p = 1$ | bundle gerbes |
| $p = 2$ | bundle $2$-gerbes |

# Tautological bundle gerbe

## Example

Let $M$ be two-connected with an integral $3$-form $H$ and fix $m \in M$. For example $M = SU(n)$. Let $Y = PM$ be the space of paths based at $m$ with endpoint evaluation as projection to $M$. If $p_1, p_2 \in Y$ have the same endpoint choose a surface $\Sigma \subset M$ spanning them, that is the boundary of $\Sigma$ is $p_1$ followed by $p_2$ with the opposite orientation.

The fibre of $P \to Y^{[2]}$ consists of all triples $(p_1, p_2, \Sigma, z)$ modulo the equivalence relation $(p_1, p_2, \Sigma, z) \sim (p_1, p_2, \Sigma', z')$ if

$$\mathrm{wzw}(\Sigma \cup \Sigma')z = \exp\Big(2\pi\,\mathrm{i} \int_B H\Big) z = z'$$

for any choice of three-manifold $B$ whose boundary is $\Sigma \cup \Sigma'$.

There is a natural bundle gerbe product.

$$(p_1, p_2, \Sigma, z) \otimes (p_2, p_3, \Sigma', z') \longmapsto (p_1, p_3, \Sigma \cup \Sigma', z z') .$$

# Real bundle gerbes

This definition of Real bundle gerbe was introduced in a more general setting by Moutuou in his thesis [3].

**Definition**

A *Real structure* on a bundle gerbe $\mathcal{G} = (P, Y)$ over $M$ is a pair of maps $(\tau_P, \tau_Y)$ where $\tau_Y : Y \to Y$ is an involution covering $\tau : M \to M$, and $\tau_P : P \to P$ is a conjugate involution covering $\tau_Y^{[2]} : Y^{[2]} \to Y^{[2]}$ and commuting with the bundle gerbe multiplication. A *Real bundle gerbe* over $M$ is a bundle gerbe $(P, Y)$ over $M$ with a Real structure.

It is straightforward to check the Real bundle gerbes pullback and you can form products and take duals.

Slightly more complicated is the question of Real trivialisation.

# Real trivialisation and Real stable isomorphism

**Definition**

Let $\mathcal{G} = (P, Y)$ be a Real bundle gerbe. A trivialisation $R \to Y$ of $(P, Y)$ is a *Real trivialisation* if $R$ has a *Real structure* which induces the Real structure on $P$ under the isomorphism $P \simeq \delta(R)$. A Real bundle gerbe is *Real trivial* if it admits a Real trivialisation.

Then we have

**Proposition**

*Real bundle gerbes are classified up to Real stable isomorphism by their Real Dixmier-Douady class in $H^2(M; \mathbb{Z}_2, \overline{\mathcal{U}(1)})$.*

We can mimic the definition of a Real bundle gerbe to define a $\mathbb{Z}_2$-(equivariant bundle gerbe) and stable isomorphism classes of the same. The useful long exact sequence then becomes

$$
\begin{tikzcd}
0 \arrow[r] & \text{Real functions} \arrow[r] & \text{functions} \arrow[r, "g \mapsto g(g\circ\tau)"] & \{h \mid h = h\circ\tau\} \arrow[dlll, out=0, in=180] \\
& \text{Real bundles} \arrow[r, "\text{forgetful}"] & \text{bundles} \arrow[r, "P \mapsto P\otimes\tau^{-1}(P)"] & \mathbb{Z}_2 \text{ bundles} \arrow[dlll, out=0, in=180] \\
& \text{Real bgs} \arrow[r, "\text{forgetful}"] & \text{bundle gerbes} \arrow[r, "\mathcal{G}\mapsto\mathcal{G}\otimes\tau^{-1}(\mathcal{G})"] & \mathbb{Z}_2 \text{ bgs} \cdots
\end{tikzcd}
$$

Again the forgetful maps forget the Real structures and in the bottom row everything is stable isomorphism classes. The connecting homomorphism can be understood geometrically.

# Real bundle gerbes over a point

Let $\{x\}$ be a space with one element. Over it we have the (trivial!) bundle $U(1) \times \{x\}$. It has two inequivalent $\mathbb{Z}_2$ structures. They are $(z, x) \mapsto (z, x)$ or $(z, x) \mapsto (\bar{z}, x)$. Hence $H^1(\{x\}; \mathbb{Z}_2, \mathcal{U}(1)) \simeq \mathbb{Z}_2$. So most of the useful long exact sequence collapses to give us

$$\mathbb{Z}_2 \simeq H^1(\{x\}; \mathbb{Z}_2, \mathcal{U}(1)) \simeq H^2(\{x\}; \mathbb{Z}_2, \overline{\mathcal{U}}(1))$$

so there are two Real bundle gerbes over a point.

## Example (The tautological bundle gerbe)

Let $M$ be two-connected. Assume that $\tau : M \to M$ has at least one fixed point $m$ and $M$ admits an integral three-form $H$ satisfying $\tau^*(H) = -H$; for example, these conditions are satisfied by the Lie group $M = SU(n)$ with $\tau(g) = g^{-1}$ and $H$ the appropriate multiple of $\mathrm{tr}((g^{-1}dg)^3)$.
Recall that we let $Y = PM$ be the space of paths based at $m$. Given a path $p$ in $M$ we can define $\tau(p)$ by $\tau(p)(t) = \tau(p(t))$ of course. But $p$ begins at $m$ a fixed point of $\tau$ so $\tau(p)$ also begins at $p$. We define a Real structure $\tau$ by the fact that

$$(p_1, p_2, \Sigma, z) \longmapsto (\tau(p_1), \tau(p_2), \tau(\Sigma), \bar{z})$$

descends through the equivalence relation to give a conjugate bundle gerbe isomorphism $P \to \tau^{-1}(P)$. The conjugate isomorphism results because $\tau^*(H) = -H$.

# Jandl gerbes

In the definition of Real bundle gerbe we used an isomorphism $\tau\colon (P, Y) \to (P^*, Y)$. It is often more natural to use *stable isomorphisms*. This could be used to motivate another notion of Real bundle gerbe. This was the approach taken by Schreiber, Schweigert and Waldorf in [6] where they called these *Jandl gerbes*.

A Real bundle gerbe is also a Jandl bundle gerbe.

We can introduce the notion of the Dixmier-Douady class of a Jandl gerbe and the corresponding idea of Jandl stable isomorphism. Then we find that

**Proposition**

*Jandl gerbes are classified up to Jandle stable isomorphism by their Real Dixmier-Douady class in $H^2(M; \mathbb{Z}_2, \overline{\mathcal{U}(1)})$.*

So Real bundle gerbes up to Real stable isomorphism are the same as Jandl bundle gerbes up to Jandle stable isomorphism.

# When does a bundle gerbe have a Real structure ?

It is worth noting that all the results we are developing here are for stable isomorphism classes. So instead of being able to answer

> *Does* $\mathcal{G} = (P, Y)$ *admit a Real (Jandl) structure ?*

we can, at best, answer

> *Is* $\mathcal{G} = (P, Y)$ *stably isomorphic to a bundle gerbe which admits a Real (Jandl) structure ?*

For example I haven't been able to show that the explicit constructions of the basic bundle gerbe on $SU(n)$ in [5] admit Real (or Jandl) structures even though I know that they are stably isomorphic to tautological bundle gerbes which admit Real structures.

# Final example

As an example of the kinds of things that can happen consider $M = S^1 \times S^2$ with the Real structure which is the product of the antipodal map on the circle and the identity map on the sphere. Then the useful long exact sequence simplifies as follows.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "a \mapsto (0,0,2a)"] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[dll, out=0, in=180] \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r, "(a,b,c,d) \mapsto d"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The connecting homomorphism is $(a, b, c) \mapsto (a, b, [c], 0)$ where $[c]$ is the class of $c \in \mathbb{Z}$ in $\mathbb{Z}_2$.

So the set of Real stable isomorphism classes of Real bundle gerbes on $S^1 \times S^2$ with this Real structure is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Whereas the set of stable isomorphism classes of bundle gerbes is $H^3(S^1 \times S^2, \mathbb{Z}) = \mathbb{Z}$.

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