Equivariant Bundle Gerbes

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Introduction

- Joint work with David Roberts, Danny Stevenson, Raymond Vozzo
- arXiv:1506.07931

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- Two things to take away from this talk:
  - weak group actions arise because of morphisms between morphisms
  - simplicial spaces are a useful way of encoding group actions

Morphisms between morphisms

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When you first learn about categories you discover the interesting fact that functors have morphisms between them called natural transformations. This means that the category of categories is an example of a 2-category, which is a category where for any two objects A and B the set $\mathrm{Mor}(A, B)$ is a category. So there are "morphisms between morphisms".

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Bundle gerbes are also an example of a 2-category. Morphisms of bundle gerbes are called *stable isomorphisms* and between two stable isomorphisms can be a morphism called a *transformation*.

This additional structure complicates the usual definition of a group action.

(Weak) group actions

Let G be a group and X a set. A (right) group action is a collection of bijections $\phi_g : X \to X$ one for each $g \in G$ such that:

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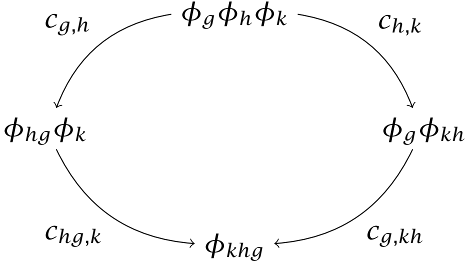
This is the idea behind the notion of a *weak group action*.

In turn these isomorphisms satisfy a *coherency condition* which is that all diagrams like the following commute:



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If G is a Lie group acting on M we say that a G-action on L is an action of G on L by bundle isomorphisms covering the action on M. So for each $g \in G$ we have:

$$\begin{array}{ccc} L & \xrightarrow{\psi_g} & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \\ m & \longmapsto & mg \end{array}$$

Of course we also want $\psi_e = \mathrm{id}_L$ and $\psi_g \psi_h = \psi_{hg}$.

Group actions and simplicial manifolds

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In other words we can regard the collection of all ψ_g as a *smooth non-vanishing section* ψ of $d_0^{-1}(L) \otimes d_1^{-1}(L)^*$. We denote this later line bundle by $\delta(L)$.

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These maps are defined as

$$\begin{aligned} d_0(m, g_0, g_1, \dots, g_p) &= (mg_0, g_1, \dots, g_p) \\ d_1(m, g_0, g_1, \dots, g_p) &= (m, g_0 g_1, g_2, \dots, g_p) \\ &\vdots \\ d_p(m, g_0, g_1, \dots, g_p) &= (m, g_0 g_1, g_2, \dots, g_{p-1} g_p) \\ d_{p+1}(m, g_0, g_1, \dots, g_p) &= (m, g_0, g_1, g_2, \dots, g_{p-1}). \end{aligned}$$

These satisfy a collection of relations which you can work out and which make it a simplicial manifold.

To remember the maps it's useful to imagine introducing the single point set $\{*\}$ on which G acts trivially on the left and replacing $M \times G^p$ by $M \times G^p \times \{*\}$. Then all the maps consist of deleting commas:

$$\begin{aligned}
d_0(m, g_0, g_1, \ldots, g_p, *) &= (mg_0, g_1, \ldots, g_p, *) \\
d_1(m, g_0, g_1, \ldots, g_p, *) &= (m, g_0g_1, g_2, \ldots, g_p, *) \\
&\vdots \\
d_{p+1}(m, g_0, g_1, \ldots, g_p, *) &= (m, g_0, g_1, \ldots, g_{p-1}, g_p *) \\
&= (m, g_0, g_1, \ldots, g_{p-1}, *).
\end{aligned}$$

Of course we don't actually do this it's just an aid to memory.

Remember we have $L \to M$ and $\delta(L) = d_0^{-1}(L) \otimes d_1^{-1}(L)^* \to M \times G$.
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Recall that we showed that the group action on L was encoded in the section ψ of $\delta(L) \to M \times G$. We can form $\delta(\psi)$ which is a section of $\delta^2(L)$ and hence a $\mathbb{C}$ valued function. A calculation shows that $\psi_g \psi_h = \psi_{hg}$ if and only if $\delta(\psi) = 1$.

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In summary: *If a Lie group G acts on a manifold M and $L \to M$ is a line bundle then an action of G on L is a section ψ of $\delta(L) \to M \times G$ satisfying $\delta(\psi) = 1$ over $M \times G^2$.*

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Notice that I am avoiding talking about non-vanishing sections. That is because bundle gerbes don't have sections but they do have trivialisations which, in the case of line bundles are the same thing.

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But now there is an additional ingredient. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two trivialisations of $\mathcal{G}$ then they differ by a line bundle A and a trivialisation of that line bundle is called a *transformation*. This is where the 2-category structure comes in. Trivialisations also behave well under pullback, tensor product and taking duals.

Equivariant bundle gerbes

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The trivialisation c is essentially the c_{gh} and the coherency condition is $\delta(c) = 1$. It can be shown that $\delta(A)$ is canonically trivial.

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> G *acts on everything commuting with all structures*

This is called a *strong group action* and it implies the previous definition which is also called a *weak group action*.

Some results

If the G action on M is such that there is a nice quotient say $M \to N$ is a principal G bundle then a line bundle $L \to M$ on which G acts descends to a quotient line bundle on N. The same is true for bundle gerbes both for strong and weak group actions.

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A bundle gerbe $\mathcal{G}$ over M is determined up to stable isomorphism by a class in $H^3(M)$ called the Dixmier-Douady class of $\mathcal{G}$. If G acts on M and we choose an action on $\mathcal{G}$ then this determines a class in the equivariant cohomology $H^3_G(M)$ which maps to the Dixmier-Douady class of $\mathcal{G}$ by the usual map

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In our preprint we determine this class for the basic bundle gerbe on $U(n)$ and the strong action of $U(n)$ covering the conjugation action on itself. This is even interesting for the case of $U(1)$ because $H^3_{U(1)}(U(1)) \neq 0$.