SUMMARY OF ALGEBRAIC TOPOLOGY 2008

Lecture 1. Monday 28th July

1. Introduction.

Discussion of what algebraic topology is good for.

2. Categories, groupoids and functors

Definition 2.1. A *category* $\mathcal{C}$ consists of two collections $\mathrm{Mor}(\mathcal{C})$ (called morphims of $\mathcal{C}$) and $\mathrm{Ob}(\mathcal{C})$ (called objects of $\mathcal{C}$) with two maps $s, t\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ called *source* and *target* satisfying the following requirements:

If $X, Y \in \mathrm{Ob}(\mathcal{C})$ denote by $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ the set of all morphisms f with $s(f) = X$ and $t(f) = Y$. Then we have a *composition*

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) &\to \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

which satisfies an *associativity* condition $(f \circ g) \circ h = f \circ (g \circ h)$ whenever the compositions are defined. Moreover for every $X \in \mathrm{Ob}(\mathcal{C})$ there is an *identity* morphism $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ which satisfies $1_Y \circ f = f \circ 1_X = f$ for every $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and every $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Note 2.1. Sometimes we denote $\mathrm{Mor}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$.

Note 2.2. If $f \in \mathrm{Mor}(X, Y)$ then we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$.

Definition 2.2. If $\mathcal{C}$ is a category a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called *invertible* if there exists $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Note 2.3. As with groups we can show that if a morphism f is invertible then the corresponding morphism g is unique. We call it the *inverse* of f and denote it by f^{-1}.

Definition 2.3. A category in which all morphisms are invertible is called a *groupoid*.

Definition 2.4. A groupoid is called *transitive* if $\mathrm{Mor}(X, Y) \neq \emptyset$ for all objects X and Y.

Proposition 2.5. *Let $\mathcal{G}$ be a groupoid. Then*

(1) For any object X, $\mathrm{Mor}_{\mathcal{G}}(X, X)$ is a group.
(2) For any morphism $f \in \mathrm{Mor}_{\mathcal{G}}(X, Y)$ the function

$$\iota_f\colon \mathrm{Mor}_{\mathcal{G}}(X,X) \to \mathrm{Mor}_{\mathcal{G}}(Y,Y)$$

defined by $\iota_f(g) = fgf^{-1}$ is an isomorphism of groups.

Corollary 2.6. *For a transistive groupoid $\mathcal{G}$ the groups $\mathrm{Mor}_{\mathcal{G}}(X, X)$ are all isomorphic.*

Lecture 2. Thursday 31st July

Definition 2.7. A *functor* F between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functions $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ and $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ such that:

(1) $F(\mathrm{Mor}_{\mathcal{C}}(X, Y)) \subseteq \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.
(2) $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.
(3) If $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ then $F(g \circ f) = F(g) \circ F(f)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Note 2.4. Sometimes we have all the conditions of a functor except that $F(\mathrm{Mor}_C(X,Y)) \subseteq \mathrm{Mor}_{\mathcal{D}}(F(Y),F(X))$ for all $X,Y \in \mathrm{Ob}(C)$ and $F(g \circ f) = F(f) \circ F(g)$. In this case we call it a *contravariant* functor and make the distinction by calling the first case above a *covariant* functor.

Lemma 2.8. *Let $F\colon C \to \mathcal{D}$ be a functor. If $f \in \mathrm{Mor}_C(X,Y)$ is a morphism in C which is invertible then $F(f)$ is invertible.*

3. Topology

3.1. Metric Spaces.

Definition 3.1. Let X be a set. Then a map $d\colon X \times X \to \mathbb{R}$ is called a *metric* on X if it satisfies:

(1) $d(x,y) \geq 0$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$.
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Note 3.1. If d is a metric on X the pair (X,d) is called a metric space.

Proposition 3.2. *Let X be any set and define*

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Then d is a metric. This metric is called the discrete metric *on X.*

Example 3.1. Let (X,d) be a metric space and $Y \subseteq X$. Define $d_Y\colon Y \times Y \to \mathbb{R}$ by restricting $d\colon X \times X \to \mathbb{R}$ to $Y \times Y \subseteq X \times X$. Then d_Y is a metric on Y. This metric is called the *subspace metric* on Y.

Definition 3.3. If (X,d) is a metric space and $x \in X$ and $\delta > 0$ then we call

$$B(X,\delta) = \{y \mid d(x,y) < \delta\}$$

the open ball around x of radius δ.

Definition 3.4. Let (X,d) be a metric space. We call a subset $U \subseteq X$ *open* if for all $x \in U$ there is a $\delta > 0$ such that $x \in B(x,\delta) \subseteq U$.

Proposition 3.5. *An open ball is an open set.*

Definition 3.6. Let (X,d) be a metric space and let $\mathcal{T}_d$ be the collection of all open subsets of X. Then:

(1) $\varnothing, X \in \mathcal{T}_d$.
(2) If U_1 and U_2 are in $\mathcal{T}_d$ then $U_1 \cap U_2 \in \mathcal{T}_d$.
(3) If U_α is in $\mathcal{T}_d$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}_d$.

3.2. Topological Spaces.

Definition 3.7. Let X be a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{T}$ is a *topology* on X if it satisfies:

(1) $\varnothing, X \in \mathcal{T}$.
(2) If U_1 and U_2 are in $\mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.
(3) If U_α is in $\mathcal{T}$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}$.

Note 3.2. If $\mathcal{T}$ is a topology we call the pair $(X,\mathcal{T})$ a *topological space* and the elements of $\mathcal{T}$ *open* subsets of X.

Definition 3.8. If X is a set then $\mathcal{T} = \mathcal{P}(X)$ is called the *discrete* topology on X.

Definition 3.9. If X is a set then $\mathcal{T} = \{\varnothing, X\}$ is called the *trivial* topology.

Proposition 3.10. *Let (X,d) be a metric space and let $\mathcal{T}_d$ be the set of all open subsets. Then $\mathcal{T}_d$ is a topology on X.*

Note 3.3. If (X,d) is a metric space we call the topology $\mathcal{T}_d$ the *metric* topology on X.

Lecture 3. Monday 4th August

Definition 3.11. If $(X, \mathcal{T})$ is a topological space and there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$ then we say that $(X, \mathcal{T})$ is *metrizable*.

Definition 3.12. We say a topological space X is *Hausdorff* if for all $x \neq y \in X$ there exist open sets U and V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proposition 3.13. *Metric spaces are Hausdorff.*

Corollary 3.14. *Not all topological spaces are metrizable.*

Definition 3.15. If $(X, \mathcal{T})$ is a topological space and $C \subseteq X$ we say that C is closed if $X - C$ is open.

Proposition 3.16. *Let $(X, \mathcal{T})$ be a topological space. Then:*

(1) $\emptyset$ and X are closed,
(2) if C_1 and C_2 are closed then $C_1 \cup C_2$ is closed, and
(3) if C_α is closed for all $\alpha \in I$ then $\cap_{\alpha \in I} C_\alpha$ is closed.

Proposition 3.17. *Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Define*

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$$

then $\mathcal{T}_Y$ is a topology on Y. This topology is called the subspace topology *on Y.*

Proposition 3.18. *Let (X, d) be a metric space and $Y \subseteq X$. Then the two topologies:*

(a) $\mathcal{T}_{d_Y}$ — the topology induced by the subspace metric, and
(b) $(\mathcal{T}_d)_Y$ — the subspace topology induced from the metric topology on X

are the same.

Proposition 3.19. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $X = X_1 \times X_2$. Define $\mathcal{T} \subseteq \mathcal{P}(X)$ by requiring that $U \in \mathcal{T}$ if for all $(x_1, x_2) \in U$ there exists U_1 open in X_1 and U_2 open in X_2 with*

$$(x_1, x_2) \in U_1 \times U_2 \subseteq U.$$

Then $\mathcal{T}$ is a topology on X. This topology is called the product topology *on $X_1 \times X_2$.*

3.3. **Continuous functions.**

Definition 3.20. Let X and Y be topological spaces. We say that $f\colon X \to Y$ is continuous if for every open subset $U \subseteq Y$ we have $f^{-1}(U) \subseteq X$ open.

Definition 3.21. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$. We say that f is continuous at x if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x' \in X$ we have

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon.$$

If f is continuous at x for all $x \in X$ we say that f is continuous.

Lecture 4. Thursday 7th August

Proposition 3.22. *Let $f\colon X \to Y$ be a function between metric spaces. Then f is continuous as a function between metric spaces if and only if it is continuous as a function between topological spaces with the metric topologies.*

Proposition 3.23. *Let $f\colon X \to Y$ be a map between topological spaces. Then f is continuous if and only if for all closed subsets $C \subseteq Y$ we have $f^{-1}(C) \subseteq X$ closed.*

Proposition 3.24. *Let X, Y and Z be topological spaces and assume $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous. Then $g \circ f\colon X \to Z$ is continuous.*

Proposition 3.25. *Let X and Y be topological spaces. If $z \in X$ then the following are continuous:*

(1) $\pi_X\colon X \times Y \to X$ defined by $\pi_X(x, y) = x$

(2) $\iota_z \colon Y \to X \times Y$ *defined by* $\iota_z(y) = (z, y)$.

Corollary 3.26.

(1) *Let* $f \colon Z \to X \times Y$ *be given by* $f(z) = (f_1(z), f_2(z))$ *where* $f_1 \colon Z \to X$ *and* $f_2 \colon Z \to Y$. *Then if* f *is continuous we have that* f_1 *and* f_2 *are continuous.*
(2) *If* $f \colon X \times Y \to Z$ *is continuous and* $x \in X$ *then* $f_x \colon Y \to Z$ *defined by* $f_x(y) = f(x, y)$ *is continuous.*

Proposition 3.27. *Let* $f \colon X \to Y_1 \times Y_2$ *be defined by* $f(x) = (f_1(x), f_2(x))$ *where* $f_1 \colon X \to Y_1$ *and* $f_2 \colon X \to Y_2$. *Then* f *continuous if and only if* f_1 *and* f_2 *are continuous.*

Proposition 3.28. *If* $Y \subseteq X$ *is given the subspace topology and* $C \subseteq Y$ *then* C *is closed in* Y *if and only if* $C = Y \cap D$ *for some* D *closed in* X.

Proposition 3.29. *If* $Y \subseteq X$ *is closed then* $C \subset Y$ *is closed in* Y *if and only if it is closed in* X.

Lemma 3.30 (Pasting Lemma)**.** *Let* $X = C \cap D$ *where* C *and* D *are closed in* X. *Let* $f \colon C \to Y$ *and* $g \colon D \to Y$ *be continuous maps into a space* Y *such that* $f(x) = g(x)$ *for all* $x \in C \cap D$. *Then* $h \colon X \to Y$ *defined by*

$$h(x) = \begin{cases} f(x) & x \in C \\ g(x) & x \in D \end{cases}$$

is a continuous map.

Definition 3.31. A continuous function $f \colon X \to Y$ between topological spaces is called a *homeomorphism* if it has a continuous inverse. Two topological spaces are called *homeomorphic* if there is homeomorphism between them.

4. Homotopy theory

4.1. **Homotopy.**

Definition 4.1. Let $f, g \colon X \to Y$ be two continuous functions between topological spaces. We say that f is *homotopic* to g if there exists a continuous function

$$H \colon [0,1] \times X \to Y$$

satisfying $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$.

Note 4.1. We denote by $H_s \colon X \to Y$ the function $H_s(x) = H(s, x)$. Note that each H_s is continuous and that $H_0 = f$ and $H_1 = g$.

Note 4.2. If f is homotopic to g we write $f \simeq g$.

Lecture 5. Monday 11th August

4.2. **Path homotopy.**

Definition 4.2. Let X be a topological space and x_0 and x_1 be points in X. Then a path in X from x_0 to x_1 is a continuous map $f \colon [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = y$.

Note 4.3. If $x_0 = x_1 = x$ then we call a path from x_0 to x_1 a *loop* in X at x.

Definition 4.3. Two paths f, f' from x_0 to x_1 are called *path homotopic* if we have a continuous map $H \colon [0,1] \times [0,1] \to X$ such that, if we define $H_s(t) = H(s, t)$, then each $H_s \colon [0,1] \to X$ is a path from x_0 to x_1 and $F_0 = f$ and $F_1 = f'$.

Proposition 4.4. *Let* X *and* Y *be topological spaces. Then homotopy is an equivalence relation on continuous functions from* X *to* Y *and path homotopy is an equivalence relation on paths from* x_0 *to* x_1 *for any points* x_0 *and* x_1 *in* X.

Note 4.4. The set of all homotopy classes of maps from X to Y is denoted $[X, Y]$.

Note 4.5. We denote the equivalence class of a path, or loop f by $[f]$.

Note 4.6. We denote by $\pi(X, x_0, x_1)$ the set of all homotopy classes of paths in X from x_0 to x_1 and denote $\pi_1(X, x) = \pi_1(X, x, x)$.

Note 4.7. We denote by $\Pi(X)$ the union

$$\Pi(X) = \bigcup_{x_0, x_1 \in X} \pi_1(X, x_0, x_1)$$

and call it the *fundamental groupoid* of X. Note that as yet it is only a set we have to show how to make it into a groupoid.

Note 4.8. Notice that if we have a homotopy H between two loops at x then each H_s is also a loop at x for every s. The set of all equivalence classes of loops at x is denoted $\pi_1(X, x)$ and called the *fundamental group* of X (at x).

Lecture 6. Monday 18th August

Definition 4.5. If f and g are paths in X we call them *composable* if $f(1) = g(0)$.

Given f and g composable paths consider the function from $[0,1]$ to X defined by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

By the Pasting Lemma this is a path from $f(0)$ to $g(1)$ called the *product* of f and g.

Lemma 4.6. *Let f and g be composable paths. If f is path homotopic to f' and g is path homotopic to g' then f' and g' are composable paths and $f * g$ is path homotopic to $f' * g'$.*

This lemma shows that there is a well-defined product

$$\pi_1(X, x_0, x_)) \times \pi_1(X, x_1, x_2) \to \pi_1(X, x_0, x_3)$$

which sends $([f],[g])$ to $([f * g])$. We denote $[f * g]$ by $[f] * [g]$.

Proposition 4.7. *Assume we have points x_0, x_1, x_2 and x_3 in X and paths f from x_0 to x_1, g from x_1 to x_2 and h from x_2 to x_3. Then*

$$(f * g) * h \simeq_p f * (h * g).$$

Lecture 7. Thursday 21st August

Proposition 4.8. *If $x \in X$ is a point in a topological space denote by e_x the constant path $e_x(t) = x$. If f is a path in X denote by f^{-1} the path $f^{-1}(t) = f(1-t)$. Then if f is a path from x_0 to x_1 we have*

(1) $e_{x_0} * f \simeq_p f$ and $f * e_{x_1} \simeq_p f$
(2) $f^{-1} * f \simeq_p e_{x_1}$ and $f * f^{-1} \simeq_p e_{x_1}$.

Proposition 4.9. *The pair $\Pi(X)$ and X define the morphisms and objects of a groupoid with the product of homotopy classes of paths as composition. The source and target maps are defined by $s([f]) = f(1)$ and $t([f]) = f(0)$. The inverse of $[\gamma]$ is $[\gamma - 1]$ where $\gamma^{-1}(t) = \gamma(1-t)$. The identity at $x \in X$ is the equivalence class of the constant path $e_x(t) = x$. This groupoid is called the homotopy groupoid of X and denoted $\Pi(X)$.*

Proposition 4.10. *If $x \in X$ then $\pi_1(X, x)$ is a group and if f is path from x_0 to x_1 the map*

$$\begin{array}{ccc} \pi_1(X, x_0) & \to & \pi_1(X, x_1) \\ {[h]} & \mapsto & [f^{-1}] * [h] * [f] \end{array} \tag{4.1}$$

is an isomorphism of groups.

Definition 4.11. We say that a topological space X is path-connected if for any $x, y \in X$ there is a path from x to y.

Proposition 4.12. *The relation 'there is a path joining x to y' is an equivalence relation on any topological space.*

Note 4.9. The equivalence classes under this relation are called the *path-components* of X.

Proposition 4.13. *A topological space X is path-connected if and only if the homotopy groupoid is transitive.*

Proposition 4.14. *If X is path connected and $x, y \in X$ then $\pi_1(X,x)$ is isomorphic to $\pi_1(X,y)$.*

Note 4.10. Because of this proposition we often drop the reference to the point x for a path connected space X and just refer to the fundamental group $\pi_1(X)$ of X.

Note 4.11. This should be compared to Lecture 2 where it is shown that this is really a result about groupoids.

Consider the category $\underline{\text{Top}}$. It is useful to modify this slightly as follows.

Definition 4.15. A *pointed topological space* is a pair (X,x) where X is a topological space and $x \in X$ is a point in X. A morphism of pointed topological spaces $(X,x) \to (Y,y)$ is a continuous map $f\colon X \to Y$ such that $f(x) = y$.

The resulting category of pointed topological spaces is denoted $\underline{\text{Top}}_*$.

Let $F\colon X \to Y$ be a continous map and let f be a path in X. Then $F \circ f$ is a path in Y. It is easy to check that if f is homotopic to f' by a homotopy H then $F \circ f$ is homotopic to $F \circ f'$ by the homotopy $F \circ H$. So we can define

$$\begin{array}{rccc} F_*\colon & \pi_1(X, x_0, x_1) & \to & \pi_1(Y, F(x_0), F(x_1)) \\ & [f] & \mapsto & [F \circ f] \end{array}$$

Note 4.12. In lectures we just talked about the case $x_0 = x_1 = x$ but the result is the same.

Proposition 4.16. *Let $F\colon X \to Y$ be a continuous function between topological spaces and f and g be composable paths in X. Then $F \circ (f * g) = (F \circ f) * (F \circ g)$. If $x \in X$ then $F_*(e_x) = e_{F(x)}$.*

Hence the pair $F_*\colon \Pi(X) \to \Pi(Y)$ and $F\colon X \to Y$ defines a functor from the fundamental groupoid of X to the fundamental groupoid of Y.

Lecture 8. Monday 25th August

Proposition 4.17. *Let $F\colon X \to Y$ and $G\colon Y \to Z$ be continuous functions between topological space. Then $(G \circ F)_* = G_* \circ F_*$. Moreover if $\mathrm{id}\colon X \to X$ is the identity map then $\mathrm{id}_* = \mathrm{id}_{\Pi(X)}$.*

Hence the pair of maps

$$\begin{array}{ccc} \mathrm{Mor}(\underline{\text{Top}}) & \to & \mathrm{Mor}(\underline{\text{Grpd}}) \\ F & \mapsto & F_* \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Ob}(\underline{\text{Top}}) & \to & \mathrm{Ob}(\underline{\text{Grpd}}) \\ X & \mapsto & \Pi(X). \end{array}$$

defines a functor $\underline{\text{Top}} \to \underline{\text{Grpd}}$ which we denote by Π.

Note 4.13. It follows that if $F\colon X \to Y$ is a continuous function then $F_*\colon \pi_1(X,x) \to \pi_1(Y, F(x))$ is a group homomorphism.

Proposition 4.18. *If $F\colon X \to Y$ is homeomorphism then $F_*\colon \pi_1(X,x) \to \pi_1(Y,F(x))$ is a group isomorphism.*

Definition 4.19. If $F\colon X \to Y$ is a continuous map between topological spaces we say it is a *homotopy equivalence* if there is a continuous map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We say two spaces X and Y are homotopy equivalent of there is a homotopy equivalence from X to Y.

Proposition 4.20. *If $F\colon X \to Y$ is a homotopy equivalence then $F_*\colon \pi_1(X,x) \to \pi_1(Y,F(x))$ is an isomorphism.*

Lemma 4.21. *(Square Lemma) Let $H\colon [0,1] \times [0,1] \to X$ be a continuous function. Define $\alpha, \beta, \gamma, \delta\colon [0,1] \to X$ by $\alpha(t) = H(t,0)$, $\beta(t) = H(1,t)$, $\gamma(t) = H(t,1)$ and $\delta(t) = H(0,t)$ for all $t \in [0,1]$. Then $\alpha * \beta \simeq_p \delta * \gamma$.*

Proposition 4.22. *Let $F, G\colon X \to Y$ be homotopic by $H\colon [0,1] \times X \to Y$. Let $f\colon [0,1] \to X$ satisfy $f(0) = x_0$ and $f(1) = x_1$. For any $x \in X$ define a path $\alpha_x(t) = H(t,x)$ then*

$$F \circ f * \alpha_{x_1} \simeq_p \alpha_{x_0} * G \circ f.$$

Proposition 4.23. *Let* $F\colon X \to X$ *be homotopic to the identify map* $\mathrm{id}_X\colon X \to X$. *Then* $F_*\colon \pi_1(X,x) \to \pi_1(X,F(x))$ *satisfies* $F \circ f \simeq_p \alpha_x^{-1} * f * \alpha_x$.

Lecture 9. Thursday 28th August

Definition 4.24. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors between categories $\mathcal{C}$ and $\mathcal{D}$. A *natural transformation* τ from F to G assigns to every object X in $\mathcal{C}$ a morphism $\tau_X\colon F(X) \to G(X)$ such that for every function $f\colon X \to Y$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\tau_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\tau_Y} & G(Y) \end{array}$$

commutes.

Note 4.14. Note that the path $\alpha_x(t) = H(x,t)$ constructed above defines a natural transformation between the two functors $F_*, G_*\colon \Pi(X) \to \Pi(Y)$ because we have the commuting diagram:

$$\begin{array}{ccc} F(x_0) & \xrightarrow{\alpha_{x_0}} & G(x_0) \\ \downarrow{\scriptstyle F_*(f)} & & \downarrow{\scriptstyle G_*(f)} \\ F(x_1) & \xrightarrow{\alpha_{x_1}} & G(x_1) \end{array}$$

Definition 4.25. A topological space is called *simply connected* if it is path connected and its fundamental group is zero.

Definition 4.26. A topological space X is called *contractible* if there exists $x_0 \in X$ such that the constant map $\hat{x}_0\colon X \to X$ defined by $\hat{x}_0(x) = x_0$ is homotopic to the identity map $\mathrm{id}_X\colon X \to X$.

Example 4.1. If $X = Y = \mathbb{R}^n$ then the identity map is constractible to the constant map to zero by $F(s,x) = sx$.

Example 4.2. Let X be a star shaped region in $\mathbb{R}^n$, that is a region $X \subseteq \mathbb{R}^n$ with a point $x \in X$ with the property that that for every other point $y \in X$ the line segment from X to y is also in X. Then X is contractible.

Note 4.15. The converse is not true. We shall see later that $\pi_1(S^2) = 0$ and S^2 is certainly path-connected but it is not contractible.

Proposition 4.27. *If* X *is contractible then* X *is simply connected.*

Definition 4.28. Let $A \subseteq X$. We call $r\colon X \to A$ a *retraction* if $r(a) = a$ for all $a \in A$ and we call A a *retract* of X.

Definition 4.29. Let $\iota\colon A \to X$ be the inclusion map then $r \circ \iota = \mathrm{id}_A$. If $\iota \circ r \simeq \mathrm{id}_X$ we call A a *deformation retract* of X and r a *deformation retraction*.

Proposition 4.30. *If* $A \subseteq X$ *is a deformation retract then* $\pi_1(A,a) \simeq \pi_1(X,a)$ *for* $a \in A$.

Lecture 10. Monday 1st September

Proposition 4.31.

$$\pi_1(X \times Y, (x,y)) \simeq \pi_1(X,x) \times \pi_1(Y,y)$$

4.3. The van Kampen Theorem.

Note 4.16. Recall that a topological space X is called *compact* if for any collection of open set $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ with $X = \bigcup_{\alpha \in I} U_\alpha$ there is a finite collection such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$. (Every open cover has a finite subcover). A subset of a $\mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine-Borel Theorem).

Lemma 4.32. *If* $[0,1] \subseteq U \cup V$ *then there exists* $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ *such that each* $[t_i, t_{i+1}]$ *is a subset of either* U *or* V.

Theorem 4.33 (Weak van Kampen Theorem). *Let* X *be a topological space with* $X = U \cup V$ *for* U, V *open and simply connected and* $U \cap V \neq \emptyset$ *and path connected. Then* X *is siomply connected.*

Example 4.3. $S^n \subseteq \mathbb{R}^{n+1}$ is simply connected for $n \geq 2$.

Definition 4.34. If G and H are groups we define the *free product*, $G * H$, as follows. Take all words $g_1 h_1 g_2 h_2 \dots g_n h_n$ with product given by justaposition. Let 1 be the empty word. Put on all the obvious relations such as $g_1 1_H g_2 = g_1 g_2$, $1_H = 1$, etc. The result is a group called the free product of G and H.

Theorem 4.35 (General van Kampen Theorem). *Let X be a topological space, U and V open sets and $X = U \cup V$. Assume, U, V and $U \cap V$ are path connected. Let $\iota_U\colon U \cap V \to U$ and $\iota_V\colon U \cap V \to V$ be the inclusion maps. Then, if $x \in U \cap V$, there is a surjective homomorphism:*

$$\pi_1(U,x) * \pi_1(V,x) \to \pi_1(X,x)$$

with kernel K generated by all elements of the form

$$(\iota_U)_*([f]) * (\iota_V)_*([f]^{-1})$$

where $[f] \in \pi_1(U \cap V, x)$. Hence

$$\pi_1(X,x) \simeq \frac{\pi_1(U,x) * \pi_1(V,x)}{K}.$$

Lecture 11. Thursday 4th September

In the next section assume all topological spaces are Hausdorff.

5. Covering spaces and the fundamental group of the circle

Proposition 5.1. *The interval $[a,b]$ is compact and path connected.*

Proposition 5.2. *If $f\colon [a,b] \to X$ is continuous, where X is a disjoint union of open sets U_α for $\alpha \in I$, then there is some $\beta \in I$ with $f([a,b]) \subseteq U_\beta$.*

Definition 5.3. Let $p\colon E \to B$ be a continuous, surjective map. We say that an open set $U \subseteq B$ is *evenly covered* if $p^{-1}(V)$ is a disjoint union of open sets V_α such that

$$p_{|V_\alpha}\colon V_a \to U$$

is a homeomorphism.

The collection $\{V_\alpha\}$ is called a *partition* of $p^{-1}(U)$ into *slices*.

Definition 5.4. Let $p\colon E \to B$ be a continuous, surjective map. We sat that p is a *covering map* and E a *covering space* if for all $x \in B$ there is an open set $U \ni x$ which is evenly covered.

Note 5.1. If $p\colon E \to B$ is a covering space and $p \in B$ then $p^{-1}(b)$ has the discrete topology as a subspace of E.

Proposition 5.5. *If $S^1 \subseteq \mathbb{R}^2$ and $p\colon \mathbb{R} \to S^1$ is the map $p(t) = (\cos(2\pi x), \sin(2\pi x))$ then p is a covering map.*

Definition 5.6. Let $p\colon E \to B$ be a continuous, surjective map. If $f\colon X \to B$ is continuous then a *lift* of f is a continuous map $\hat{f}\colon X \to E$ with $p \circ \hat{f} = f$.

Proposition 5.7 (Path lifting property). *Let $p\colon E \to B$ be a covering space with $p(e) = b$ where $e \in E$. Let $f\colon [0,1] \to B$ with $f(0) = b$. Then f has a unique lift $\hat{f}\colon [0,1] \to E$ with $\hat{f}(0) = e$.*

Lecture 12. Monday 8th September

Lemma 5.8. *Let $\{U_\alpha \mid \alpha \in I\}$ be an open cover of $[0,1] \times [0,1]$ and let $s \in [0,1]$. There there exists $\epsilon > 0$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for all $i = 0, \dots, n-1$ there is some $\alpha \in I$ with $(s-\epsilon, s+\epsilon) \times [t_i, t_{i+1}] \subseteq U_\alpha$.*

Proposition 5.9 (Covering homotopy property). *Let $p\colon E \to B$ be a covering space with $p(e_0) = b_0$ where $e_0 \in E$. If $f, g\colon [0,1] \to B$ are paths from b_0 to b_1 and $H\colon [0,1] \times [0,1] \to B$ is a path homotopy from f to g then H has a unique lift $\hat{H}\colon [0,1] \times [0,1] \to E$ such that $\hat{H}(0,0) = e_0$. Moreover $\hat{H}$ is a path homotopy between $\hat{f}$ and $\hat{g}$.*

Proposition 5.10. *Let $p\colon E \to B$ be a covering space with $p(e_0) = b_0$ for $e_0 \in E$. let $F = p^{-1}(b_0)$. Then there is a map $\alpha\colon \pi_1(B, e_0) \to p^{-1}(b_0)$ such that if we consider*

$$\pi_1(E, e_0) \overset{p_*}{\to} \pi_1(B, b_0) \overset{\alpha}{\to} F$$

such that

(1) p_* is injective,
(2) α is surjective, and
(3) $\alpha^{-1}(e_0) = \operatorname{im}(p_*)$.

Note 5.2. This is a statement about *sets*. To show what the fundamental *group* is we need to work a little harder in particular cases. Usually F has some natural structure as a group and we prove that α is a homomorphism.

Proposition 5.11. $\pi_1(S^1) = \mathbb{Z}$

Lecture 13. Thursday 11th September

6. Applications of homotopy theory

6.1. **Brouwer Fixed Point Theorem.** Define the two dimensional disk $D^2 = \{(x, y) \mid x^2 + y^2 \le 1\}$. Notice that $S^1 \subseteq D^2$.

Theorem 6.1 (Brouwer Fixed Point Theorem). *If $f\colon D^2 \to D^2$ is a continuous function then there is a $d \in D^2$ with $f(d) = d$. (i.e f has a fixed point)*

Lemma 6.2. *There is no retraction $r\colon D^2 \to S^1$.*

6.2. **Degree of a map $f\colon S^1 \to S^1$.**

Definition 6.3. Consider $f\colon S^1 \to S^1$ and define $F(t) = f(\cos(2\pi t), \sin(2\pi t))$ where $F\colon [0,1] \to S^1$. Lift F to $\hat{F}\colon [0,1] \to \mathbb{R}$ and define

$$\deg(f) = \hat{F}(1) - \hat{F}(0).$$

This is called the *degree* (or winding number) of F.

Proposition 6.4. *The degree of f is an integer and independent of the choice of the lift $\hat{F}$. It also depends only on the homotopy class of f.*

Note 6.1. If the loops in question are differentiable we can construct the degree by the integral

$$\frac{1}{2\pi} \int \frac{f'(t)}{f(t)} dt.$$

which calculates the winding number of f.

Theorem 6.5. *Let $p(z) = z^n + c_1 z^{n-1} + \cdots + c_n$ be a polynomial of degree n. Then there is a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

7. Singular Homology

7.1. **Definitions.**

Definition 7.1. If $q \ge 0$ we define the *standard q-simplex* $\triangle_q \subset \mathbb{R}^{q+1}$ by

$$\triangle_q = \{(t^0, \dots, t^q) \mid \sum_{i=0}^{q} t^i = 1, t^i \ge 0 \ \forall i = 0, \dots, q\} \subseteq \mathbb{R}^q$$

Definition 7.2. Given a topological space X a *(singular) q-simplex in X* σ is a continuous map $\sigma\colon \triangle_q \to X$. We denote the set of all singular q-simplices in X by $\triangle_q(X)$.

Lecture 14. Monday 15th September

Definition 7.3. Define $S_q(X)$ to be the free abelian group generated by $\triangle_q(X)$. Call the elements of $S_q(X)$ *(singular) q-chains*.

Definition 7.4. For $q > 0$ define $F_q^i\colon \triangle_{q-1} \to \triangle_q$, $0 \le i \le q$ (the ith *face map*) by

$$F_q^i(t^0, \dots, t^{q-1}) = (t^0, t^1, \dots, t^{i-1}, 0, t^i, \dots, t^{q-1}).$$

Definition 7.5. If $\sigma \in \triangle_q(X)$ for $q > 0$ we define $\partial\sigma \in S_{q-1}(X)$ by

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma \circ F_q^i.$$

We extend this to a homomorphism $\partial\colon S_q(X) \to S_{q-1}(X)$ called the *boundary map*. If $c \in S_0(X)$ we define $\partial c = 0$.

Lecture 15. Thursday 18th September

Proposition 7.6. $\partial\partial = 0$

Lemma 7.7. *For $j < i$ we have $F_q^i F_{q-1}^j = F_q^j F_{q-1}^{i-1}$.*

Definition 7.8. If $q > 0$ we define

$$Z_q = \ker(\partial\colon S_q(X) \to S_{q-1}(X)) \subseteq S_q(X), \quad \text{and} \quad B_q = \operatorname{im}(\partial\colon S_{q+1}(X) \to S_q(X)) \subseteq S_q(X).$$

We also let $B_0 = \operatorname{im}(\partial\colon S_1(X) \to S_0(X)) \subseteq S_0(X)$ and $Z_0 = S_0(X)$. Call the elements of $Z_q(X)$ *cycles* and the elements of $B_q(X)$ *boundaries*. For each $q \ge 0$ we define

$$H_q(X) = Z_q/B_q$$

the qth *homology group* of X.

Example 7.1. Let X be a single point then

$$H_q(X) = \begin{cases} \mathbb{Z} & \text{q} = 0 \\ 0 & \text{q} > 0 \end{cases}$$

Proposition 7.9. *Let the topological space X have path components $X_1, \dots, X_r$ then for all $q \ge 0$ we have*

$$H_q(X) \simeq H_q(X_1) \oplus \cdots \oplus H_q(X_r).$$

Proposition 7.10. *Let the topological space X have path components $X_1, \dots, X_r$ then $H_0(X) \simeq \mathbb{Z}^r$.*

Lecture 16. Thursday 9th October

If G is a group we define the *commutator subgroup* of G to be the subgroup $[G, G]$ generated by

$$\{ghg^{-1}h^{-1} \mid g, h \in G\}.$$

We have that $[G, G]$ is a normal subgroup and that $G/[G, G]$ is abelian. In fact if N is a normal subgroup of G with G/N abelian then $[G, G] \subseteq N$.

Proposition 7.11 (Not proved)**.**

$$H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

Structure of $H_p(X)$. For most spaces you are likely to meet $H_p(X)$ is finitely generated. The structure theory of finitely generated abelian groups is as follows. If A is a finitely generated abelian group we define the torsion subgroup of A by

$$\operatorname{Tor}(A) = \{a \in A \mid \exists n \text{ s.t. } a^n = 0\}.$$

Then there is a unique r such that

$$A \simeq \operatorname{Tor}(A) \oplus \mathbb{Z}^r$$

and $d_1, \dots, d_n$ such that

$$\operatorname{Tor}(A) \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_n}.$$

The unique number r is called the *rank* of A.

Definition 7.12. If X is a topological space with finitely generated homology groups then the rank of $H_q(X)$ is called the q-th *Betti number* of X denoted $\beta_q(X)$.

Definition 7.13. If X is a topological space with a finite number of non-zero homology groups all of which are finitely generated then the Euler characteristic of X is

$$\chi(X) = \sum_q \beta_q s(X).$$

Note 7.1. If we replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition of homology then we get $H_q(X, \mathbb{R})$ which are vector spaces. If the homology is finitely generated then $H_q(X, \mathbb{R}) = \mathbb{R}^{\beta_q(X)}$.

7.2. Some homological algebra.

Definition 7.14. A *chain complex* $G_\bullet = \{G_p, \partial_p\}$ of abelian groups is a collection of abelian groups and homomorphims between them

$$\dots \overset{\partial_{p+1}}{\to} G_p \overset{\partial_p}{\to} G_{p-1} \overset{\partial_{p-1}}{\to} \dots \overset{\partial_2}{\to} G_1 \overset{\partial_1}{\to} G_0$$

such that $\partial_{p-1} \circ \partial_p = 0$ for $p > 1$.

Definition 7.15. If $G_\bullet = \{G_p, \partial_p\}$ is a chain complex we define its homology groups by

$$H_p(G_\bullet) = \frac{\ker \partial_p \colon G_p \to G_{p+1}}{\operatorname{im} \partial_{p-1} \colon G_{p-1} \to G_p}$$

for $p \geq 1$ and

$$H_0(G_\bullet) = \frac{G_0}{\operatorname{im} \partial_1 \colon G_1 \to G_0}.$$

Definition 7.16. If $G_\bullet = \{G_p, \partial_p\}$ and $K_\bullet = \{K_p, \partial_p\}$ are chain complexes a morphism $f_\bullet \colon G_\bullet \to K_\bullet$ is a family of maps $f_p \colon G_p \to K_p$ for $p = 0, 1, 2, \dots$ such that $f_{p-1}\partial_p = \partial_p f_p$ for all $p = 1, 2, \dots$.

Note 7.2. Chain complexes and their morphisms form a category $\underline{\text{ChComp}}$

Definition 7.17. Let $H_\bullet = \{H_0, H_1, H_2, \dots\}$ and $L_\bullet = \{L_0, L_1, L_2, \dots\}$ be two sequences of abelian groups. A morphism from $H_\bullet$ to $L_\bullet$ is a sequence of maps $f_\bullet = \{f_0, f_1, f_2, \dots\}$ such that each $f_p \colon H_p \to L_p$ is a morphism of abelian groups.

Note 7.3. Sequences of abelian groups and their morphisms define a category $\underline{\text{SAbGrp}}$.

Note 7.4. A chain map $f_\bullet \colon G_\bullet \to K_\bullet$ induces a morphism $H_\bullet(f_\bullet)$ of the sequences of abelian groups $H_\bullet(G_\bullet)$ and $H_\bullet(K_\bullet)$ and hence a functor $H \colon \underline{\text{ChComp}} \to \underline{\text{SAbGrp}}$.

Note 7.5. Singular homology of topological spaces is the composition of two functors:

$$\begin{array}{ccccc} \underline{\text{Top}} & \overset{S}{\to} & \underline{\text{ChComp}} & \overset{H}{\to} & \underline{\text{SAbGrp}} \\ X & \mapsto & S_\bullet = \{S_p(X), \partial_p\} & \mapsto & \{H_p(X)\} \end{array}$$

Many of the differences between homology and homotopy come from the properties of the functor H the study of which is called *homological algebra*.

Lecture 17. Monday 13th October

7.3. Functorial properties of S.

Definition 7.18. If $f \colon X \to Y$ is a continuous function and $\sigma \in \triangle_q(X)$ then $f \circ \sigma \in \triangle_q(Y)$. Extending this to a homomorphism defines $S_q(f) \colon S_q(X) \to S_q(Y)$ by $S_q(f)(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma (\sigma \circ f)$.

Lemma 7.19. *If $f \colon X \to Y$ is a continuous function then for all $q \geq 0$:*

a) $S_q(f)$ *is a chain map, i.e* $\partial S_{q+1}(f) = S_q(f)\partial$
b) $S_q(\mathrm{id}_X) = \mathrm{id}_{S_q(X)}$
c) $S_q(f \circ g) = S_q(f) \circ S_q(g)$.

Note 7.6. This shows that $X \mapsto S_q(X)$ and $f \mapsto S_q(f)$ defines a functor $S \colon \underline{\text{Top}} \to \underline{\text{ChComp}}$.

Definition 7.20. If $f\colon X \to Y$ is a continuous function we define $H_q(f)\colon H_q(X) \to H_q(Y)$ by $H_q(f) = H_q(S_\bullet(f))$ so that

$$H_q(f)([c]) = [S_q(f)(c)]$$

for all $[c] \in H_q(X)$.

Lemma 7.21. *Denote by $\delta_q \in S_q(\triangle_q)$ the identity map $\mathrm{id}_{\triangle_q}\colon \triangle_q \to \triangle_q$ considered as a q-simplex. Then if $\sigma\colon \triangle_q \to X$ we have $S_q(\sigma)(\delta_q) = \sigma \in S_q(X)$.*

8. The homotopy invariance theorem

Theorem 8.1. *If $f, g\colon X \to Y$ are homotopic then*

$$H_q(f) = H_q(g)\colon H_q(X) \to H_q(Y)$$

for all $q \geq 0$.

Lecture 18. Thursday 16th October

Lecture 19. Monday 20th October

Proposition 8.2. *If X and Y are homotopy equivalent spaces then $H_q(X) \simeq H_q(Y)$ for all $q \geq 0$.*

Corollary 8.3. *If X is contractible then*

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q > 0 \end{cases}$$

Definition 8.4. Let $f_\bullet$ and $g_\bullet$ be chain maps between chain complexes $G_\bullet$ and $K_\bullet$. We say that f and g are *homotopic* if for every $q \geq 0$ there is a homomorphism $P_q\colon G_q \to K_{q+1}$ such that

$$P_{q-1}\partial + \partial P_q = f_q - g_q$$

for all $q \geq 0$. We call such a sequence of maps $P_\bullet$ a *homotopy* from $f_\bullet$ to $g_\bullet$.

Proposition 8.5. *If $f_\bullet$ and $g_\bullet$ are homotopic chain maps between chain complexes $G_\bullet$ and $K_\bullet$ then $H_q(f_\bullet) = H_q(g_\bullet)$ for all $q \geq 0$.*

Proposition 8.6. *If f and g are homotopic maps from X to Y then $S_\bullet(f)$ and $S_\bullet(g)$ are homotopic chain maps.*

9. The Mayer-Vietoris sequence

Definition 9.1. A sequence of abelian groups and homomorphism

$$\cdots \xrightarrow{f_{i-2}} A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is called *exact* at A_i if $\ker f_i = \operatorname{im} f_{i-1}$. It is called *exact* if it is exact at all A_i.

Note 9.1. If A is an abelian group and 0 denotes the zero group then there is only one homomorphism $0 \to A$ and only one homomorphism $A \to 0$.

Definition 9.2. A short exact sequence of abelian groups is an exact sequence of the form

$$0 \to A \to B \to C \to 0.$$

Proposition 9.3. *If A and B are abelian groups then*

(a) $0 \to A \xrightarrow{f} B$ *is exact if and only if f is injective.*
(b) $A \xrightarrow{f} B \to 0$ *is exact if and only if f is surjective.*
(c) $0 \to A \xrightarrow{f} B \to 0$ *is exact if and only if f is bijective.*

Lecture 20. Thursday 23rd October

Lemma 9.4 (The five lemma). *Let*

$$\begin{array}{ccccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A' & \xrightarrow[g_1]{} & B' & \xrightarrow[g_2]{} & C' & \xrightarrow[g_3]{} & D' & \xrightarrow[g_4]{} & E' \end{array}$$

be a commutative diagram of abelian groups and homomorphism where both the horizontal rows are exact. If α, β, δ and ϵ are isomorphisms then so also is γ.

Definition 9.5. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of a topological space X. We say a q-simplex is $\mathcal{U}$-small if the image of σ is contained in some element of $\mathcal{U}$. Denote by $\triangle_q^{\mathcal{U}}(X) \subseteq \triangle_q(X)$ the set of all $\mathcal{U}$-small q-simplices and by $S_q^{\mathcal{U}}(X) \subseteq S_q(X)$ the subgroup of chains formed from $\mathcal{U}$ small q-simplices.

Note 9.2. $S_\bullet^{\mathcal{U}}(X)$ is a subcomplex of $S_\bullet(X)$ and hence we can form its homology denoted $H_q^{\mathcal{U}}(X)$. The chain map $S_\bullet^{\mathcal{U}}(X) \subseteq S_\bullet(X)$ induces a map $H_q^{\mathcal{U}}(X) \to H_q(X)$ for all $q \geq 0$.

Proposition 9.6 (Not proved). *$H_q^{\mathcal{U}}(X) \to H_q(X)$ is an isomorphism for all $q \geq 0$.*

Let $X = U \cup V$ where U and V are open sets and let $\mathcal{U} = \{U, V\}$. Define $\alpha\colon S_q(U\cap V) \to S_q(U) \oplus S_q(V)$ by $\alpha(a) = (a, -a)$ and define $\beta\colon S_q(U) \oplus S_q(V) \to S_q^{\mathcal{U}}(M)$ by $\beta(c,d) = c + d$. Then we have

Proposition 9.7. *The homomorphisms α and β are chain maps and*

$$0 \to S_q(U\cap V) \xrightarrow{\alpha} S_q(U) \oplus S_q(V) \xrightarrow{\beta} S_q^{\mathcal{U}}(M) \to 0$$

is a short exact sequence for all $q \geq 0$.

Lecture 22. Monday 27th October

Proposition 9.8. *Let*

$$0 \to A_q \xrightarrow{\alpha} B_q \xrightarrow{\beta} C_q \to 0$$

be a short exact sequence of chain maps for all $q \geq 0$. Then for every $q \geq 1$ there is a homomorphism (called the connecting homomorphism*) $\delta\colon H_q(C_\bullet) \to H_{q-1}(A_\bullet)$ such that*

$$\cdots \xrightarrow{\delta} H_1(A_\bullet) \xrightarrow{H_1(\alpha)} H_1(B_\bullet) \xrightarrow{H_1(\beta)} H_1(C_\bullet) \xrightarrow{\delta} H_0(A_\bullet) \xrightarrow{H_0(\alpha)} H_0(B_\bullet) \xrightarrow{H_0(\beta)} H_0(C_\bullet) \longrightarrow 0$$

is an exact sequence (called the long exact homology sequence).

Proposition 9.9. *Let $X = U \cup V$ where U and V are open sets then there is an exact sequence*

$$\cdots \xrightarrow{\delta} H_1(U\cup V) \xrightarrow{H_1(\alpha)} H_1(U) \oplus H_1(V) \xrightarrow{H_1(\beta)} H_1(M) \xrightarrow{\delta} H_0(U\cup V) \xrightarrow{H_0(\alpha)} H_1(U) \oplus H_0(V) \xrightarrow{H_0(\beta)} H_0(M) \longrightarrow 0$$

(called the Mayer-Vietoris sequence for U and V).

9.1. Homology of spheres.

Proposition 9.10.

$$H_q(S^1) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} & q = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 9.11.

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} & q = n \\ 0 & \textit{otherwise} \end{cases}$$

Lecture 23. Thursday 30th October

Concluding remarks and review.