1. Answer all questions true or false. Give a short justification of each answer: this might just be 'result proved in lectures'.
   (i) Every topological space has a metric giving rise to the topoloygy.
   (ii) $\mathcal{T} = \{\emptyset, \{a,b\}, \{b,c\}, X\}$ is a topology on the set $X = \{a,b,c\}$.
   (iii) If X has the discrete topology and $f\colon X \to \mathbb{R}^{17}$ then f is continuous.
   (iv) The open sets in a metric space form a topology.
   (v) $\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$.
   (vi) If x and y are points in a topological space X then $\pi_1(X,x)$ is isomorphic to $\pi_1(X,y)$.
   (vii) $S^1 \times S^1 \times S^1$ is homotopy equivalent to S^3.
   (viii) The sphere S^{123} is contractible.
   (ix) A contractible space is simply-connected.
   (x) There exists a topological space X with $H_{55}(X) = \mathbb{Z}$.

2. (a) (i) Define what a category is.
   (ii) Define what a groupoid is.
   (iii) If $\mathcal{G}$ is a groupoid and $\mathrm{Mor}_{\mathcal{G}}(X,Y) \neq \emptyset$ show that $\mathrm{Mor}_{\mathcal{G}}(X,X)$ and $\mathrm{Mor}_{\mathcal{G}}(Y,Y)$ are isomorphic groups. (You may assume known that in a groupoid $\mathrm{Mor}_{\mathcal{G}}(X,X)$ is a group.)

   (b) (i) Define what it means for $\mathcal{T}$ to be a topology on a set X.
   (ii) Let X be a set and define $\mathcal{T}$ to be the set of all subsets $U \subseteq X$ which satisfy either $U = \emptyset$ or $X - U$ is a finite set of points (remember that 0 is a finite number). Show that $\mathcal{T}$ is a topology on X.

3. (a) Let X be a topological space.
   (i) If f and g are paths from x to y define what it means for f and g to be path homotopic.
   (ii) Show that path homotopy is an equivalence relation on paths from x to y.
   (iii) Let e_x be the constant path $e_x(t) = x$. If f is a path from x to y show that $e_x \star f$ is path homotopic to f.

   (b) Prove the Brouwer fixed point theorem that states that any continuous function $f\colon D \to D$ has a fixed point when $D = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$. You may assume that $\pi_1(S^1) \neq 0$.

4. (a) (i) Define what it means for a continuous function $p\colon E \to B$ to be a covering map.

   (ii) State the path lifting property and the covering homotopy property for a covering map $p\colon E \to B$.

   (iii) Let $p\colon \mathbb{R} \to S^1$ be the map $p(t) = (\cos(t), \sin(t))$. Assuming that this is a covering map calculate $\pi_1(S^1)$. State carefully any results you use from lectures.

   (iv) What can you conclude about $H_1(S^1)$?

   (b) (i) Define what it means for a sequence of abelian groups and homomorphisms

$$0 \to G \xrightarrow{\alpha} H \xrightarrow{\beta} K \to 0$$

   to be a short exact sequence of abelian groups.

   (ii) Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{f_1} & H & \xrightarrow{f_2} & K & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & G' & \xrightarrow[g_1]{} & H' & \xrightarrow[g_2]{} & K' & \longrightarrow & E' \end{array}$$

   where both the horizontal rows are short exact sequences. If α and γ are isomorphisms show that γ is an isomorphism. (Don't just quote the Five Lemma.)

5. (a) Define carefully the pth homology group $H_p(X)$ of a topological space X. Include a definition of the boundary map.

   (b) If $f, g\colon X \to Y$ are homotopic continuous functions between topological spaces what can we conclude about $H_p(f), H_p(g)\colon H_p(X) \to H_p(Y)$? (Don't prove this result.)

   (c) Using the result from (b) show that $H_p(\mathbb{R}^n) = 0$ except when $p = 0$ when it equals $\mathbb{Z}$.

   (d) Consider a topological space X which is a union of two open sets U and V. Define the Mayer-Vietoris long exact sequence in homology and explain what all the maps are.

   (e) Using the Mayer-Vietoris sequence calculate the homology of the two-sphere S^2.