Algebraic Topology IV 2008

Assignment 3 – Due Friday 7th November

1. Consider a commutative diagram of abelian groups

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{f_1} & H & \xrightarrow{f_2} & K & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \longrightarrow & G' & \xrightarrow[g_1]{} & H' & \xrightarrow[g_2]{} & K' & \longrightarrow & E'
\end{array}
$$

where both the horizontal rows are short exact sequences. If α and γ are isomorphisms show that γ is an isomorphism. (Don't just quote the Five Lemma.)

2. Consider X_r a 'bouquet' of r circles. That is r circles all joined at one point. Use the Mayer-Vietoris sequence to calculate the homology of X_r. You can indicate by drawing pictures plausible deformation retractions.

3. Consider $\mathbb{C}P_n$ the space of all lines in $\mathbb{C}^{n+1}$. We denote the line through the non-zero vector $(z_0, z_1, \dots, z_n)$ by $[z_0, z_1, \dots, z_n]$. Notice that if $\lambda \neq 0$ then $[\lambda z_0, \lambda z_1, \dots, \lambda z_n] = [z_0, z_1, \dots, z_n]$ and that there is a well-defined, surjective, function $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P_n$ which sends a non-zero vector to the line containing it.

(a) Let $U_n \subset \mathbb{C}P_n$ and $H_{n-1} \subset \mathbb{C}P_n$ be defined by

$$U_n = \{[z_0, \dots, z_n] \mid z_n \neq 0\}$$

and

$$H_{n-1} = \{[z_0, \dots, z_{n-1}, 0] \mid (z_0, \dots, z_n) \in \mathbb{C}^n\}$$

Show that $\mathbb{C}P_n$ is the disjoint union of U_n and H_{n-1}.

(b) Show that there are well-defined bijections $U_n \to \mathbb{C}^n$ defined by

$$[z_0, \dots, z_n] \mapsto \left(\frac{z_0}{z_n}, \dots, \frac{z_{n-1}}{z_n}\right)$$

and $\mathbb{C}P_{n-1} \to H$ defined by

$$[z_0, \dots, z_{n-1}] \mapsto [z_0, \dots, z_{n-1}, 0].$$

(c) Let $V = \mathbb{C}P_n - \{[0, 0, \dots, 1]\}$ and show that H is a deformation retract of V. I haven't explained what the topology is on $\mathbb{C}P_n$ so I will accept any well-defined homotopy that looks plausibly continuous. For those familiar with the notion of quotient topology the topology on $\mathbb{C}P_n$ is the quotient topology for the function $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P_n$. The sets V and U_n are open.

(d) Using V and U_n and induction calculate the homology of $\mathbb{C}P_n$ using the Mayer-Vietoris sequence. You may assume that H with the subspace topology is homeomorphic to $\mathbb{C}P_{n-1}$.