Algebraic Topology IV 2008

Assignment 1 – Due Thursday 11th September

1. Consider the categories Set and Vec of all sets and all vectors spaces respectively. Define a function F on objects by $F(X) = \mathrm{Map}(X, \mathbb{R})$. Find an action of F on maps of sets $f : X \to Y$ making it a functor. Is this a contravariant or covariant functor ?

2. Let $X = (\mathbb{Z}_2)^r$ where $\mathbb{Z}_2 = \{0, 1\}$. Define $d : X \times X \to \mathbb{R}$ by letting $d(x, y)$ be the number of places at which the binary string $x = (x_1, \ldots, x_r)$ differs from the binary string $y = (y_1, \ldots, y_r)$. Show that d is a metric.

This metric is called the Hamming distance and is used in coding theory. The idea is that if a binary string x is transmitted down a noisy channel it will be corrupted by having some 0's and 1's randomly become 1's and 0's. If there is only a small amount of noise the received string should be close to the transmitted string relative to the Hamming distance and this idea can be used to construct error-correcting codes.

3. Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. We say that $\mathcal{B}$ is a basis for a topology on X if:

(a) for all $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

(b) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Show that the set of all open balls in a metric space X is a basis for a topology on X. If B is a basis for a topology on X we define $\mathcal{T}_\mathcal{B} \subseteq \mathcal{P}(X)$ by saying that $U \in \mathcal{T}_\mathcal{B}$ if for all $x \in U$ there is a $B \in \mathcal{B}$ with $x \in B \subseteq U$. Show that $\mathcal{T}_\mathcal{B}$ is a topology on X. We call $\mathcal{T}_\mathcal{B}$ the topology generated by $\mathcal{B}$.

4. If f and g are path-homotopic paths in a topological space X show that f^{-1} and g^{-1} are path-homotopic.

5. Let X be a topological space containing points x, y, z and w. Let α be a path from x to y, β a path from y to z and γ a path from z to w. Choose numbers $0 < a < b < 1$. Define a path $\alpha \star \beta \star \gamma$ from x to w by

$$\alpha \star \beta \star \gamma = \begin{cases} \alpha(t/a) & 0 \le t \le a \\ \beta((t-a)/(b-a)) & a \le t \le b \\ \gamma((t-b)/(1-b)) & b \le t \le 1 \end{cases}$$

Show that $\alpha \star \beta \star \gamma$ is a continuous path from x to w. Show that the homotopy class of $\alpha \star \beta \star \gamma$ is independent of a and b.

6. If X and Y are topological spaces we denote by $[X, Y]$ the set of all homotopy classes of continuous maps from X to Y. Denote also the unit interval $[0, 1]$ by I.

(a) Show that for any topological space X the set $[X, I]$ has only one element.

(b) Show that for any path-connected, topological space Y the set $[I, Y]$ has only one element.

Come and ask if you want a hint.