SUMMARY OF ALGEBRAIC TOPOLOGY 2006

Note: This is as summary of the course as I expect it to look as of 2006/7/24 if we don’t do de Rham cohomology. It will no doubt change along the way in which case I will hand out an updated summary.

1. INTRODUCTION.

Discussion of what algebraic topology is good for.

2. CATEGORIES, GROUPOIDS AND FUNCTORS

Definition 2.1. A category \( C \) consists of a pair of sets \( \text{Mor}(C) \) and \( \text{Ob}(C) \) with two maps \( s, t : \text{Mor}(C) \to \text{Ob}(C) \) called source and target satisfying the following requirements:

If \( X, Y \in \text{Ob}(C) \) denote by \( \text{Mor}(X, Y) \) the set of all morphisms \( f \) with \( s(f) = X \) and \( t(f) = Y \). Then we have a composition

\[
\text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)
\]

\[
(f, g) \mapsto g \circ f
\]

which satisfies an associativity condition \((f \circ g) \circ h = f \circ (g \circ h)\) whenever the compositions are defined. Moreover for every \( X \in \text{Ob}(C) \) there is an identity morphism \( 1_X \) which satisfies \( 1_Y \circ f = f \circ 1_X = f \) for every \( f \in \text{Mor}(X, Y) \) and every \( X, Y \in \text{Ob}(C) \).

Definition 2.2. If \( C \) is a category a morphism \( f \in \text{Mor}(X, Y) \) is called invertible if there exists \( g \in \text{Mor}(Y, X) \) such that \( g \circ f = 1_X \) and \( f \circ g = 1_Y \).

Note 2.1. As with groups we can show that if a morphism \( f \) is invertible then the corresponding morphism \( g \) is unique. We call it the inverse \((f^{-1})\) of \( f \).

Definition 2.3. A category in which all morphisms are invertible is called a groupoid.

Proposition 2.4. Let \( C \) be a groupoid. Then

1. For any object \( X \), \( \text{Mor}(X, X) \) is a group.
2. For any morphism \( f \in \text{Mor}(X, Y) \) the function

\[
\iota_f : \text{Mor}(X, X) \to \text{Mor}(Y, Y)
\]

defined by \( \iota_f(g) = fgf^{-1} \) is an isomorphism of groups.

Definition 2.5. A groupoid is called transitive if \( \text{Mor}(X, Y) \neq \emptyset \) for all objects \( X \) and \( Y \).

Corollary 2.6. For a transitive groupoid the groups \( \text{Mor}(X, X) \) are all isomorphic.

Definition 2.7. A functor \( F \) between two categories \( C \) and \( D \) is a pair of functions \( F : \text{Mor}(C) \to \text{Mor}(D) \) and \( F : \text{Ob}(C) \to \text{Ob}(D) \) such that:

1. \( F(\text{Mor}(X, Y)) \subseteq \text{Mor}(F(X), F(Y)) \) for all \( X, Y \in \text{Ob}(C) \).
2. \( F(1_X) = 1_{F(X)} \) for all \( X \in \text{Ob}(C) \).
3. If \( f \in \text{Mor}(X, Y) \) and \( g \in \text{Mor}(Y, Z) \) then \( F(g \circ f) = F(g) \circ F(f) \) for all \( X, Y \in \text{Ob}(C) \).

Note 2.2. Sometimes we have all the conditions of a functor except that \( F(g \circ f) = F(f) \circ F(g) \). In this case we call it a contravariant functor and make the distinction by calling the case above a covariant functor.

Lemma 2.8. Let \( F : C \to D \) be a functor. If \( f \in \text{Mor}(X, Y) \) is a morphism in \( C \) which is invertible then \( F(f) \) is invertible.
3. Topology

3.1. Metric Spaces.

**Definition 3.1.** Let $X$ be a set. Then a map $d : X \times X \to \mathbb{R}$ is called a **metric** on $X$ if it satisfies:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Note 3.1.** If $d$ is a metric on $X$ the pair $(X, d)$ is called a metric space.

**Proposition 3.2.** Let $X$ be any set and define

$$d(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Then $d$ is a metric. This metric is called the **discrete metric** on $X$.

**Proposition 3.3.** Let $(X, d)$ be a metric space and $Y \subset X$. Define $d_Y : Y \times Y \to \mathbb{R}$ by restricting $d : X \times X \to \mathbb{R}$ to $Y \times Y \subset X \times X$. Then $d_Y$ is a metric on $Y$. This metric is called the **subspace metric** on $Y$.

**Definition 3.4.** If $(X, d)$ is a metric space and $x \in X$ and $\delta > 0$ then we call

$$B(x, \delta) = \{y \mid d(x, y) < \delta\}$$

the open ball around $x$ of radius $\delta$.

**Definition 3.5.** Let $(X, d)$ be a metric space. We call a subset $U \subset X$ **open** if for all $x \in U$ there is a $\delta > 0$ such that $x \in B(x, \delta) \subset U$.

**Definition 3.6.** Let $(X, d)$ be a metric space and let $T_d$ be the collection of all open subsets of $X$. Then:

1. $\emptyset, X \in T_d$.
2. If $U_1$ and $U_2$ are in $T_d$ then $U_1 \cap U_2 \in T_d$.
3. If $U_\alpha$ is in $T_d$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $T_d$.

3.2. Topological Spaces.

**Definition 3.7.** Let $X$ be a set and $\mathcal{T} \subset \mathcal{P}(X)$ be a collection of subsets of $X$. We say that $\mathcal{T}$ is a **topology** on $X$ if it satisfies:

1. $\emptyset, X \in \mathcal{T}$.
2. If $U_1$ and $U_2$ are in $\mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.
3. If $U_\alpha$ is in $\mathcal{T}$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}$.

**Note 3.2.** If $\mathcal{T}$ is a topology we call the pair $(X, \mathcal{T})$ a **topological space** and the elements of $\mathcal{T}$ **open** subsets of $X$.

**Definition 3.8.** If $X$ is a set then $\mathcal{T} = \mathcal{P}(X)$ is called the **discrete** topology on $X$.

**Definition 3.9.** If $X$ is a set then $\mathcal{T} = \{\emptyset, X\}$ is called the **trivial** topology.

**Proposition 3.10.** Let $(X, d)$ be a metric space and let $T_d$ be the set of all open subsets. Then $T_d$ is a topology on $X$.

**Note 3.3.** If $(X, d)$ is a metric space we call the topology $T_d$ the **metric** topology on $X$.

**Definition 3.11.** If $(X, \mathcal{T})$ is a topological space and there exists a metric $d$ on $X$ such that $\mathcal{T} = T_d$ then we say that $(X, \mathcal{T})$ is **metrizable**.

**Definition 3.12.** We say a topological space $X$ is **Hausdorff** if for all $x \neq y \in X$ there exist open sets $U$ and $V$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Proposition 3.13.** Metric spaces are Hausdorff.

**Corollary 3.14.** Not all topological spaces are metrizable.

**Definition 3.15.** If $(X, \mathcal{T})$ is a topological space and $C \subset X$ we say that $C$ is closed if $X - C$ is open.

**Proposition 3.16.** Let $(X, \mathcal{T})$ be a topological space. Then:

1. $\emptyset$ and $X$ are closed,
Proposition 3.17. Let \((X, \mathcal{T})\) be a topological space and \(Y \subset X\). Define
\[
\mathcal{Y} = \{U \cap Y \mid U \in \mathcal{T}\}
\]
then \(\mathcal{Y}\) is a topology on \(Y\). This topology is called the subspace topology on \(Y\).

Proposition 3.18. Let \((X, d)\) be a metric space and \(Y \subset X\). Then the metric topology \(\mathcal{T}_{d_Y}\) on \(Y\) determined by the subspace metric coincides with the subspace topology on \(Y\) determined by the metric topology on \(X\).

Proposition 3.19. Let \((X_1, \mathcal{T}_1)\) and \((X_3, \mathcal{T}_2)\) be topological spaces and let \(X = X_1 \times X_2\). Define \(\mathcal{T} \subset \mathcal{P}(X)\) by requiring that \(U \in \mathcal{T}\) if for all \((x_1, x_2) \in U\) there exists \(U_1\) open in \(X_1\) and \(U_2\) open in \(X_2\) with \((x_1, x_2) \in U_1 \times U_2 \subset U\).

Then \(\mathcal{T}\) is a topology on \(X\). This topology is called the product topology on \(X_1 \times X_2\).

3.3. Continuous functions.

Definition 3.20. Let \(X\) and \(Y\) be topological spaces. We say that \(f: X \to Y\) is continuous if for every open subset \(U\) of \(Y\) we have \(f^{-1}(U) \subset X\) open.

Definition 3.21. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(f: X \to Y\). We say that \(f\) is continuous if for all \(x \in X\) and for all \(\epsilon > 0\) there is a \(\delta > 0\) such that \(f(B(x, \delta)) \subset B(y, \epsilon)\).

Proposition 3.22. Let \(f: X \to Y\) be a function between metric spaces. Then \(f\) is continuous as a function between metric spaces if and only if it is continuous as a function between topological spaces with the metric topologies.

Proposition 3.23. Let \(f: X \to Y\) be a map between topological spaces. Then \(f\) is continuous if and only if for all closed subsets \(C \subset Y\) we have \(f^{-1}(C) \subset X\) closed.

Proposition 3.24. Let \(X, Y\) and \(Z\) be topological spaces and assume \(f: X \to Y\) and \(g: Y \to Z\) are continuous. Then \(g \circ f: X \to Z\) is continuous.

Proposition 3.25. Let \(X\) and \(Y\) be topological spaces. If \(z \in X\) then the following are continuous:
1. \(\pi_X: X \times Y \to X\) defined by \(\pi_X(x, y) = x\)
2. \(\iota_Z: Y \to X \times Y\) defined by \(\iota_Z(y) = (z, y)\).

Corollary 3.26. If \(f: X \times Y \to Z\) is continuous and \(x \in X\) then \(f_X: Y \to Z\) defined by \(f_X(y) = f(x, y)\) is continuous.

Proposition 3.27. Let \(f: X \to Y_1 \times Y_2\) be defined by \(f(x) = (f_1(x), f_2(x))\) where \(f_1: X \to Y_1\) and \(f_2: X \to Y_2\). Then \(f\) continuous if and only if \(f_1\) and \(f_2\) are continuous.

Lemma 3.28 (Pasting Lemma). Let \(X = C \cap D\) where \(C\) and \(D\) are closed in \(X\). Let \(f: C \to Y\) and \(g: D \to Y\) be continuous maps into a space \(Y\) such that \(f(x) = g(x)\) for all \(x \in C \cap D\). Then \(h: X \to Y\) defined by
\[
h(x) = \begin{cases} 
  f(x) & x \in C \\
  g(x) & x \in D
\end{cases}
\]
is a continuous map.

4. Homotopy theory

4.1. Homotopy.

Definition 4.1. Let \(f, g: X \to Y\) be two continuous functions between topological spaces. We say that \(f\) is homotopic to \(g\) if there exists a continuous function
\[
H: [0, 1] \times X \to Y
\]
satisfying \(H(0, x) = f(x)\) and \(H(1, x) = g(x)\) for all \(x \in X\).

Note 4.1. We denote by \(H_s: X \to Y\) the function \(H_s(x) = H(s, x)\). Note that each \(H_s\) is continuous and that \(H_0 = f\) and \(H_1 = g\).

Note 4.2. If \(f\) is homotopic to \(g\) we write \(f \simeq g\).

Proposition 4.2. Homotopy is an equivalence relation on continuous functions from \(X\) to \(Y\).
4.2. Path homotopy.

**Definition 4.3.** Let $X$ be a topological space and $x$ and $y$ be points in $X$. Then a path in $X$ from $x$ to $y$ is a continuous map $\gamma: [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

**Note 4.3.** If $x = y$ then we call the path a *loop* in $X$ at $x$.

**Definition 4.4.** Two paths $\gamma, \gamma'$ are called path homotopic if we have a continuous map $H: [0, 1] \times [0, 1] \to X$ such that, if we define $F_s(t) = F(s, t)$, then each $F_s: [0, 1] \to X$ is a path from $x$ to $y$ and $F_0 = \gamma$ and $F_1 = \gamma'$.

**Proposition 4.5.** Path homotopy is an equivalence relation on the set of all paths from $x$ to $y$.

**Note 4.4.** We denote the equivalence class of a path, or loop $y$ by $[y]$.

**Note 4.5.** Notice that if we have a homotopy between two loops at $x$ then each $F_s$ is also a loop at $x$ for every $s$. The set of all equivalence classes of loops at $x$ is denoted $\pi_1(X, x)$ and called the *fundamental group* of $X$ (at $x$).

**Definition 4.6.** If $\alpha$ and $\beta$ are paths in $X$ we call them *composable* if $\alpha(1) = \beta(0)$.

Given $\gamma$ and $\beta$ composable paths consider the function from $[0, 1]$ to $X$ defined by

$$\alpha \ast \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

By the Pasting Lemma this is a path from $\alpha(0)$ to $\beta(1)$ called the product of $\alpha$ and $\beta$. We call $\alpha \ast \beta$ the path product of $\alpha$ and $\beta$.

**Lemma 4.7.** If $\alpha$ and $\beta$ are as above and $\alpha$ is homotopic to $\alpha'$ and $\beta$ to $\beta'$ then $\alpha \ast \beta$ is homotopic to $\alpha' \ast \beta'$.

This lemma shows that there is a well-defined product of homotopy classes of paths and loops defined by $[\alpha][\beta] = [\alpha \ast \beta]$. Denote by $\Pi(X)$ the set of all paths in $X$ and define $s, t: \Pi(X) \to X$ by $s([\gamma]) = y(0)$ and $t([\gamma]) = y(1)$.

**Proposition 4.8.** The pair $\Pi(X)$ and $X$ define the morphisms and objects of a groupoid with the path product. The inverse of $[\gamma]$ is $[\gamma^{-1}]$ where $\gamma^{-1}(t) = y(1 - t)$. The identity at $x \in X$ is the equivalence class of the constant path $e_x(t) = x$.

This groupoid is called the homotopy groupoid of $X$ and denoted $\Pi(X)$.

**Definition 4.9.** We say that a topological space $X$ is path-connected if for any $x, y \in X$ there is a path from $x$ to $y$.

**Proposition 4.10.** The relation ‘there is a path joining $x$ to $y$’ is an equivalence relation on any topological space.

**Note 4.6.** The equivalence classes under this relation are called the *path-components* of $X$.

**Proposition 4.11.** A topological space $X$ is path-connected if and only if the homotopy groupoid is transitive.

**Note 4.7.** The group of all morphisms beginning and ending at $x$ in $\Pi(X)$ is denoted $\pi_1(X, x)$ and called the *fundamental group* of $X$. It is the set of all path-homotopy classes of loops as $x$ with the path product.

We can apply the results from groupoids as follows: If $\alpha$ is a path from $x$ to $y$ and $\gamma$ is a loop at $x$ then we can define a loop at $y$ by $(\alpha^{-1} \gamma) \alpha$ where $\alpha^{-1}(t) = \alpha(1 - t)$. We define a map

$$I_{[\alpha]}: \pi_1(X, x) \to \pi_1(X, y)$$

by $I_{[\alpha]}([\gamma]) = [(\alpha^{-1} \gamma) \alpha] = [\alpha^{-1} (y \alpha)]$. and it follows as before that:

**Proposition 4.12.** The map $I_{[\alpha]}$ is a group isomorphism and $(I_{[\alpha]})^{-1} = I_{[\alpha^{-1}]}$.

**Note 4.8.** Because of this proposition we often drop the reference to the point $x$ for a path connected space $X$ and just refer to the fundamental group $\pi_1(X)$ of $X$. 

4.3. Contractible maps.

**Definition 4.13.** If the identity map on \( X \) is homotopic to a constant map then we call \( X \) contractible.

**Example 4.1.** If \( X = Y = \mathbb{R}^n \) then the identity map is contractible to the constant map to zero by \( F(s, x) = sx \).

**Example 4.2.** Let \( X \) be a star shaped region in \( \mathbb{R}^n \), that is a region \( X \subset \mathbb{R}^n \) with a point \( x \in X \) with the property that for every other point \( y \in X \) the line segment from \( X \) to \( y \) is also in \( X \). Then \( X \) is contractible.

**Definition 4.14.** A topological space is called simply connected if it is path connected and its fundamental group is zero.

**Proposition 4.15.** A contractible space is simply-connected.

**Note 4.9.** The converse is not true. We shall see later that \( \pi_1(S^2) = 0 \) and \( S^2 \) is certainly path-connected but it is not contractible.

If \( f : X \to Y \) is a continuous map then we can define a map \( \pi_1(f) = f_* \) from \( \pi_1(X, x) \) to \( \pi_1(Y, f(x)) \) as follows. Let \( \gamma \) be a loop at \( x \) then \( f \circ \gamma \) is a loop at \( f(x) \). It is easy to check that if \( \gamma \) is homotopic to \( \gamma' \) then \( f \circ \gamma \) is homotopic to \( f \circ \gamma' \) and so we can define \( f_*([\gamma]) = [f \circ \gamma] \). It is also easy to check that \( f_* \) is a group homomorphism.

Let \( f \) and \( g \) be homotopic continuous maps from a space \( X \) to a space \( Y \). Let \( F : [0, 1] \times X \to Y \) be a homotopy. Then we have \( f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) \) and \( g_* : \pi_1(X, x) \to \pi_1(Y, g(x)) \). Define \( \alpha(t) = F(t, x) \), then \( \alpha \) is a path from \( f(x) \) to \( g(x) \). Recall the definition of \( I_{[\alpha]}: \pi_1(Y, f(x)) \to \pi_1(Y, g(x)) \) from above. Then we have

**Proposition 4.16.** With the notation as in the preceding discussion we have \( I_{[\alpha]} \circ f_* = g_* \).

**Definition 4.17.** A map \( f : X \to Y \) is called a homotopy equivalence if there is a map \( g : Y \to X \) such that \( f \circ g \) is homotopic to \( \text{id}_Y \) and \( g \circ f \) is homotopic to \( \text{id}_X \).

**Definition 4.18.** Two spaces are called homotopy equivalent if there is a homotopy equivalence between them.

**Corollary 4.19.** If two space are homotopy equivalent then they have isomorphic fundamental groups.

**Example 4.3.** A contractible space is homotopic to a point, that is to a space with only one element.

**Example 4.4.** The space \( \mathbb{R}^2 - \{0\} \) is homotopy equivalent to \( S^1 \).

4.4. The fundamental group of the circle. We define a continuous map \( p : \mathbb{R} \to S^1 \) by \( p(x) = (\cos(x), \sin(x)) \). It follows from elementary calculus that we can cover \( S^1 \) by open sets \( U_i \) such that there are continuous maps \( s_i : U_i \to \mathbb{R} \) such that \( p(s_i(y)) = y \) for all \( y \in S^1 \). We have two important results.

**Proposition 4.20.** (Path lifting property.) Let \( \gamma \in S^1 \) and \( x \in \mathbb{R} \) with \( p(x) = \gamma \). Let \( \gamma \) be a loop at \( x \) then there is a unique continuous map \( \hat{\gamma} : [0, 1] \to \mathbb{R} \) such that \( p \circ \hat{\gamma} = \gamma \) and \( \hat{\gamma}(0) = x \).

**Note 4.10.** We call \( \hat{\gamma} \) a lift of \( \gamma \) or we say that it covers \( \gamma \).

**Proposition 4.21.** (Covering homotopy property.) Let \( \gamma \in S^1 \) and \( x \in \mathbb{R} \) with \( p(x) = \gamma \). Let \( \gamma \) and \( \hat{\gamma} \) be loops at \( x \) and \( F : [0, 1] \times [0, 1] \to S^1 \) be a homotopy from \( \gamma \) to \( \hat{\gamma} \). Let \( \hat{\gamma} \) be a lift of \( \gamma \) with \( \gamma(0) = x \). Then there is a unique lift of \( F \) to a map \( \hat{F} : [0, 1] \times [0, 1] \to \mathbb{R} \) such that \( p \circ \hat{F} = F \) and \( F(0, t) = \hat{\gamma}(t) \) for all \( t \).

It follows that if \( \gamma \) is a loop in \( S^1 \) then \( (1/2\pi)(\hat{\gamma}(0) - \hat{\gamma}(1)) \) is an integer that depends only on the homotopy class of \( \gamma \). We have

**Proposition 4.22.** The map \( [\gamma] \to \frac{1}{2\pi}(\hat{\gamma}(0) - \hat{\gamma}(1)) \) defines an isomorphism between \( \pi_1(S^1, x) \) and \( \mathbb{Z} \).

**Note 4.11.** We call this integer the degree or winding number of \( \gamma \).

**Note 4.12.** If the loops in question are differentiable we can construct the winding number by the integral \( \frac{1}{2\pi} \int \frac{1}{y} \frac{dy}{dt} dt \).

**Corollary 4.23.** The fundamental group of \( \mathbb{R}^2 - \{0\} \) is \( \mathbb{Z} \).

**Theorem 4.24** (Brouwer fixed point theorem). If \( f : D \to D \) is a continuous map of the disk \( D = \{ x \in \mathbb{R}^2 \mid |x|^2 \leq 1 \} \) to itself then \( f \) has a fixed point, that is there is an \( x \in D \) with \( f(x) = x \).
4.5. Fundamental group of a product.

**Proposition 4.25.** Let $F : X \to Y \times Z$ be a function between topological spaces. Then it defines functions $f : X \to Y$ and $g : X \to Z$ by $F(x) = (f(x), g(x))$. Moreover any pair of functions $f$ and $g$ defines a function $F$ in this manner. In such a situation $F$ is continuous if and only if the two functions $f$ and $g$ are continuous.

With this result it easy to prove:

**Proposition 4.26.** The fundamental group of a product $X \times Y$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

**Example 4.5.** The fundamental group of a torus $S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

4.6. **Van Kampen theorem.** If $G$ and $H$ are two groups we define the free product $G \ast H$ to consist of all finite ‘words’

$$g_1h_1g_2h_2 \ldots g_kh_k$$

subject to the obvious identifications if some of the $g_i$ or $h_i$ are the identity. For example if $e_H$ is the identity in $H$ then $g_1e_Hg_2h_2 = (g_1g_2)h_2$. We define a product on $G \ast H$ by juxtaposing words and simplifying if necessary. For example if we justapose $gh$ and $e_2h^{-1}g'$ the result would be $(gh)(e_2h^{-1}g') = gg'$.

If $S \subset G$ is a subset of a group define $\langle S \rangle$ to be the normal subgroup generated by $S$ that is the smallest normal subgroup containing $S$.

Let $X = U \cup V$ where $U$ and $V$ are open and $U \cap V$ is path-connected. Define $\iota_U$ and $\iota_V$ to be the inclusion maps from $U \cap V$ into $U$ and $V$ respectively. Then we have

**Theorem 4.27** (Van Kampen theorem (not proved)). *In the situation above the homotopy group of $X$ is*

$$\pi_1(X) = \frac{\pi_1(U) \ast \pi_1(V)}{\langle \{\iota_U([y])\iota_V([y]) \mid [y] \in \pi_1(U \cap V) \rangle \rangle}.$$  

**Corollary 4.28** (Weak Van Kampen theorem). If $\pi_1(U) = \pi_1(V) = 0$ and $U \cap V$ is path connected then $\pi_1(U \cup V) = 0$.

**Corollary 4.29.** If $X = U \cup V$ and $U \cap V$ is simply connected then $\pi_1(X) = \pi_1(U) \ast \pi_1(V)$.

**Proposition 4.30.** The fundamental group of the $n$ sphere for $n > 1$ is 0.

**Proposition 4.31.** The fundamental group of the plane with $r$ points removed is the free group on $r$ generators that is the free product of $r$ copies of $\mathbb{Z}$.

4.7. **Final thoughts about homotopy theory.** In this section I discussed some other aspects of homotopy theory that aren’t for examination. I talked about the higher homotopy groups $\pi_n(X)$, and why they are abelian. Discussed suspension and how it shows that $\pi_k(S^n)$ is $\mathbb{Z}$ if $k = n$ and zero if $0 < k < n$. Drew the Hopf fibration. Also explained the Poincare conjecture. Also talked about the fundamental group of a Riemann surface and showed how to cut it up to make a polygon.

5. Homology of geometric complexes

5.1. Geometric complexes and polyhedra.

**Definition 5.1.** A set of $k + 1$ points $a_0, \ldots, a_{k+1}$ in $\mathbb{R}^n$ is called geometrically independent if they lie in no $k - 1$ dimensional hyperplane.

**Definition 5.2.** Let $a_0, \ldots, a_k$ be a geometrically independent set of points in $\mathbb{R}^n$. The $k$ dimensional geometric simplex or $k$-simplex spanned by them is

$$< a_0 \ldots a_k > = \left\{ \sum_{i=0}^{k} \lambda_ia_i \mid \sum_{i=0}^{k} \lambda_i = 1, \quad 0 \leq \lambda_i \leq 1 \right\}.$$  

**Note 5.1.** The numbers $\lambda_0, \ldots, \lambda_k$ are called the barycentric co-ordinates of $x = \sum_{i=0}^{k} \lambda_ia_i$. The subset of $< a_0 \ldots a_k >$ consisting of all points with positive barycentric co-ordinates is called the open $k$-simplex.

**Note 5.2.** The $a_i$ are called the vertices of the $k$-simplex $< a_0 \ldots a_k >$.

**Definition 5.3.** A simplex $\sigma_k$ is called a face of a simplex $\sigma_n$ if every vertex of $\sigma_k$ is a vertex of $\sigma_n$.

**Definition 5.4.** Two simplices $\sigma_n$ and $\sigma_m$ are called properly joined if $\sigma_n \cap \sigma_m = \emptyset$ or $\sigma_n \cap \sigma_m$ is a face of each of $\sigma_n$ and $\sigma_m$. 

Definition 5.5. A geometric complex (or simplicial complex) is a finite family \( K \) of geometric simplices which are properly joined and such that any face of a simplex in \( K \) is also in \( K \).

The union of all the simplices in \( K \) with the subspace topology from \( \mathbb{R}^n \) is denoted \(|K|\) and called the geometric carrier of \( K \) or the polyhedron associated to \( K \).

Definition 5.6. Let \( X \) be a topological space. If \( X \) is homeomorphic to \(|K|\) then \( X \) is called triangulable and \( K \) is called a triangulization of \( K \).

Definition 5.7. The closure of a simplex \( \text{Cl} \sigma_k \) is the complex consisting of \( \sigma^k \) and all its faces.

Definition 5.8. If \( K \) is a complex the \( r \)-skeleton is the set of all simplices of dimension less than or equal to \( k \).

5.2. Orientation of geometric complexes.

Definition 5.9. An orientation of a complex is a choice of ordering up to an even permutation.

Note 5.3. We extend the notation introduced so that \( \langle a_0 \ldots a_k \rangle = + \langle a_0 \ldots a_k \rangle \) denotes the oriented simplex with orientation determined by the ordering \( a_0 \ldots a_k \) and \( - \langle a_0 \ldots a_k \rangle \) denotes the simplex with the opposite orientation.

Definition 5.10. A complex \( K \) is called oriented if each simplex in \( K \) is oriented.

Definition 5.11. Let \( K \) be an oriented complex. Let \( \sigma^{p+1} \) and \( \sigma^p \) be two simplices in \( K \) of dimension \( p+1 \) and \( p \) respectively. Define \([\sigma^{p+1}, \sigma^p]\) as follows. If \( \sigma^p \) is not a face of \( \sigma^{p+1} \) then \([\sigma^{p+1}, \sigma^p]=0\). If \( \sigma^p=\langle a_0 \ldots a_p \rangle \) and \( v \) is the additional vertex in \( \sigma^{p+1} \) then \( \sigma^{p+1} = [\sigma^{p+1}, \sigma^p] \langle v a_0 \ldots a_p \rangle \).

Theorem 5.12. Let \( \sigma^p, \sigma^{p-2} \) be a \( p \) simplex and a \( p-2 \) simplex in an oriented complex \( K \). Then
\[
\sum_{\sigma^{p-1} \in K} [\sigma^p, \sigma^{p-1}] [\sigma^{p-1}, \sigma^{p-2}] = 0.
\]

5.3. Chains, cycles, boundaries and homology groups.

Definition 5.13. Let \( K \) be an oriented complex. A \( p \)-chain is a formal linear finite linear combination \( \sum m_i \sigma^p_i \) where the \( m_i \) are integers and the \( \sigma^p_i \) are from \( K \). We denote by \( C_p(K) \) the set of all \( p \)-chains.

Definition 5.14. We define the boundary map \( \partial: C_p(K) \to C_{p-1}(K) \) by
\[
\partial(\sum g_i \sigma^p_i) = \sum g_i [\sigma^p_i, \sigma^{p-1}] \sigma^{p-1}.
\]

Theorem 5.15.
\[
\partial^2 = 0
\]

Definition 5.16. A \( p \)-cycle is a \( p \)-chain in the kernel of \( \partial \) which we denote by \( Z_p(K) \). A \( p \)-boundary is an element of the image of \( \partial \) which we denote by \( B_p(K) \). We say two cycles are homologous if they differ by a boundary. We define \( H_p(K) = Z_p(K)/B_p(K) \) to be the \( p \)th homology group of \( K \).

Theorem 5.17. The homology groups are independent of the choice of orientation on \( K \).

Definition 5.18. Two simplices \( \sigma_0 \) and \( \sigma_1 \) in a complex \( K \) are said to be combinatorially connected if there exist vertices \( a_0, \ldots, a_p \) in \( K \) with \( a_0 \) a vertex of \( \sigma_0 \), \( a_p \) a vertex of \( \sigma_1 \) and such that \( \langle a_0, a_1, a_2, \ldots, a_p \rangle \) are all in \( K \).

Note 5.4. Combinatorial connectedness is an equivalence relation and the equivalence classes are called the combinatorial components of \( K \).

Proposition 5.19. A complex \( K \) is combinatorially connected if and only if \(|K|\) is path connected.

Theorem 5.20. If \( K \) is a complex then \( H^0(K) = \mathbb{Z}^d \) where \( d \) is the number of combinatorial components of \( K \).

5.4. Final comments. The last result shows that \( H^0(K) \) depends only on the topology of \(|K|\). Much more is true, if \( K \) and \( L \) are complexes and \(|K|\) and \(|L|\) are homeomorphic then \( H^p(K) = H^p(L) \) for all \( p \).