SUMMARY OF ALGEBRAIC TOPOLOGY 2006

Note: This is as summary of the course as I expect it to look as of 2006/7/24 if we don't do de Rham cohomology. It will no doubt change along the way in which case I will hand out an updated summary.

1. Introduction.

Discussion of what algebraic topology is good for.

2. Categories, groupoids and functors

Definition 2.1. A *category* C consists of a pair of sets $\mathrm{Mor}(C)$ and $\mathrm{Ob}(C)$ with two maps $s, t\colon \mathrm{Mor}(C) \to \mathrm{Ob}(C)$ called *source* and *target* satisfying the following requirements:

If $X, Y \in \mathrm{Ob}(C)$ denote by $\mathrm{Mor}(X,Y)$ the set of all morphisms f with $s(f) = X$ and $t(f) = Y$. Then we have a *composition*

$$\begin{aligned} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) &\to \mathrm{Mor}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

which satisfies an *associativity* condition $(f \circ g) \circ h = f \circ (g \circ h)$ whenever the compositions are defined. Moreover for every $X \in \mathrm{Ob}(C)$ there is an *identity* morphism 1_X which satisfies $1_Y \circ f = f \circ 1_X = f$ for every $f \in \mathrm{Mor}(X,Y)$ and every $X, Y \in \mathrm{Ob}(C)$.

Definition 2.2. If C is a category a morphism $f \in \mathrm{Mor}(X,Y)$ is called *invertible* if there exists $g \in \mathrm{Mor}(Y,X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Note 2.1. As with groups we can show that if a morphism f is invertible then the corresponding morphism g is unique. We call it the *inverse* (f^{-1}) of f.

Definition 2.3. A category in which all morphisms are invertible is called a *groupoid*.

Proposition 2.4. *Let C be a groupoid. Then*

1. *For any object X, $\mathrm{Mor}(X,X)$ is a group.*
2. *For any morphism $f \in \mathrm{Mor}(X,Y)$ the function*

$$\iota_f\colon \mathrm{Mor}(X,X) \to \mathrm{Mor}(Y,Y)$$

defined by $\iota_f(g) = fgf^{-1}$ is an isomorphism of groups.

Definition 2.5. A groupoid is called *transitive* if $\mathrm{Mor}(X,Y) \neq \emptyset$ for all objects X and Y.

Corollary 2.6. *For a transistive groupoid the groups $\mathrm{Mor}(X,X)$ are all isomorphic.*

Definition 2.7. A *functor* F between two categories C and $\mathcal{D}$ is a pair of functions $F\colon \mathrm{Mor}(C) \to \mathrm{Mor}(\mathcal{D})$ and $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(\mathcal{D})$ such that:

1. $F(\mathrm{Mor}(X,Y)) \subset \mathrm{Mor}(F(X), F(Y))$ for all $X, Y \in \mathrm{Ob}(C)$.
2. $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{Ob}(C)$.
3. If $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Y,Z)$ then $F(g \circ f) = F(g) \circ F(f)$ for all $X, Y \in \mathrm{Ob}(C)$.

Note 2.2. Sometimes we have all the conditions of a functor except that $F(g \circ f) = F(f) \circ F(g)$. In this case we call it a *contravariant* functor and make the distinction by calling the case above a *covariant* functor.

Lemma 2.8. *Let $F\colon C \to \mathcal{D}$ be a functor. If $f \in \mathrm{Mor}(X,Y)$ is a morphism in C which is invertible then $F(f)$ is invertible.*

3. Topology

3.1. Metric Spaces.

Definition 3.1. Let X be a set. Then a map $d\colon X \times X \to \mathbb{R}$ is called a *metric* on X if it satisfies:
(1) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Note 3.1. If d is a metric on X the pair (X,d) is called a metric space.

Proposition 3.2. *Let X be any set and define*

$$d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Then d is a metric. This metric is called the discrete metric *on X.*

Proposition 3.3. *Let (X,d) be a metric space and $Y \subset X$. Define $d_Y\colon Y \times Y \to \mathbb{R}$ by restricting $d\colon X \times X \to \mathbb{R}$ to $Y \times Y \subset X \times X$. Then d_Y is a metric on Y. This metric is called the* subspace metric *on Y.*

Definition 3.4. If (X,d) is a metric space and $x \in X$ and $\delta > 0$ then we call

$$B(X,\delta) = \{y \mid d(x,y) < \delta\}$$

the open ball around x of radius δ.

Definition 3.5. Let (X,d) be a metric space. We call a subset $U \subset X$ *open* if for all $x \in U$ there is a $\delta > 0$ such that $x \in B(x,\delta) \subset U$.

Definition 3.6. Let (X,d) be a metric space and let $\mathcal{T}_d$ be the collection of all open subsets of X. Then:
(1) $\emptyset, X \in \mathcal{T}_d$.
(2) If U_1 and U_2 are in $\mathcal{T}_d$ then $U_1 \cap U_2 \in \mathcal{T}_d$.
(3) If U_α is in $\mathcal{T}_d$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}_d$.

3.2. Topological Spaces.

Definition 3.7. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{T}$ is a *topology* on X if it satisfies:
(1) $\emptyset, X \in \mathcal{T}$.
(2) If U_1 and U_2 are in $\mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.
(3) If U_α is in $\mathcal{T}$ for all $\alpha \in I$ then $\cup_{\alpha \in I} U_\alpha$ is in $\mathcal{T}$.

Note 3.2. If $\mathcal{T}$ is a topology we call the pair $(X,\mathcal{T})$ a *topological space* and the elements of $\mathcal{T}$ *open* subsets of X.

Definition 3.8. If X is a set then $\mathcal{T} = \mathcal{P}(X)$ is called the *discrete* topology on X.

Definition 3.9. If X is a set then $\mathcal{T} = \{\emptyset, X\}$ is called the *trivial* topology.

Proposition 3.10. *Let (X,d) be a metric space and let $\mathcal{T}_d$ be the set of all open subsets. Then $\mathcal{T}_d$ is a topology on X.*

Note 3.3. If (X,d) is a metric space we call the topology $\mathcal{T}_d$ the *metric* topology on X.

Definition 3.11. If $(X,\mathcal{T})$ is a topological space and there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$ then we say that $(X,\mathcal{T})$ is *metrizable.*

Definition 3.12. We say a topological space X is Hausdorff if for all $x \neq y \in X$ there exist open sets U and V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proposition 3.13. *Metric spaces are Hausdorff.*

Corollary 3.14. *Not all topological spaces are metrizable.*

Definition 3.15. If $(X,\mathcal{T})$ is a topological space and $C \subset X$ we say that C is closed if $X - C$ is open.

Proposition 3.16. *Let $(X,\mathcal{T})$ be a topological space. Then:*
(1) *$\emptyset$ and X are closed,*

(2) *if* C_1 *and* C_2 *are closed then* $C_1 \cup C_2$ *is closed, and*
(3) *if* C_α *is closed for all* $\alpha \in I$ *then* $\cap_{\alpha \in I} C_\alpha$ *is closed.*

Proposition 3.17. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subset X$. *Define*

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$$

then $\mathcal{T}_Y$ *is a topology on* Y. *This topology is called the* subspace topology *on* Y.

Proposition 3.18. *Let* (X, d) *be a metric space and* $Y \subset X$. *Then the metric topology* $\mathcal{T}_{d_Y}$ *on* Y *determined by the subspace metric coincides with the subspace topology on* Y *determined by the metric topology on* X.

Proposition 3.19. *Let* $(X_1, \mathcal{T}_1)$ *and* $(X_3, \mathcal{T}_2)$ *be topological spaces and let* $X = X_1 \times X_2$. *Define* $\mathcal{T} \subset \mathcal{P}(X)$ *by requiring that* $U \in \mathcal{T}$ *if for all* $(x_1, x_2) \in U$ *there exists* U_1 *open in* X_1 *and* U_2 *open in* X_2 *with*

$$(x_1, x_2) \in U_1 \times U_2 \subset U.$$

Then $\mathcal{T}$ *is a topology on* X. *This topology is called the* product topology *on* $X_1 \times X_2$.

3.3. **Continuous functions.**

Definition 3.20. Let X and Y be topological spaces. We say that $f: X \to Y$ is continuous if for every open subset U of Y we have $f^{-1}(U) \subset X$ open.

Definition 3.21. Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$. We say that f is continuous if for all $x \in X$ and for all $\epsilon > 0$ there is a $\delta > 0$ such that $f(B(x, \delta)) \subset B(y, \epsilon)$.

Proposition 3.22. *Let* $f: X \to Y$ *be a function between metric spaces. Then* f *is continuous as a function between metric spaces if and only if it is continuous as a function between topological spaces with the metric topologies.*

Proposition 3.23. *Let* $f: X \to Y$ *be a map between topological spaces. Then* f *is continuous if and only if for all closed subsets* $C \subset Y$ *we have* $f^{-1}(C) \subset X$ *closed.*

Proposition 3.24. *Let* X, Y *and* Z *be topological spaces and assume* $f: X \to Y$ *and* $g: Y \to Z$ *are continuous. Then* $g \circ f: X \to Z$ *is continuous.*

Proposition 3.25. *Let* X *and* Y *be topological spaces. If* $z \in X$ *then the following are continuous:*

(1) $\pi_X: X \times Y \to X$ *defined by* $\pi_X(x, y) = x$
(2) $\iota_z: Y \to X \times Y$ *defined by* $\iota_z(y) = (z, y)$.

Corollary 3.26. *If* $f: X \times Y \to Z$ *is continuous and* $x \in X$ *then* $f_x: Y \to Z$ *defined by* $f_x(y) = f(x, y)$ *is continuous.*

Proposition 3.27. *Let* $f: X \to Y_1 \times Y_2$ *be defined by* $f(x) = (f_1(x), f_2(x))$ *where* $f_1: X \to Y_1$ *and* $f_2: X \to Y_2$. *Then* f *continuous if and only if* f_1 *and* f_2 *are continuous.*

Lemma 3.28 (Pasting Lemma). *Let* $X = C \cap D$ *where* C *and* D *are closed in* X. *Let* $f: C \to Y$ *and* $g: D \to Y$ *be continuous maps into a space* Y *such that* $f(x) = g(x)$ *for all* $x \in C \cap D$. *Then* $h: X \to Y$ *defined by*

$$h(x) = \begin{cases} f(x) & x \in C \\ g(x) & x \in D \end{cases}$$

is a continuous map.

4. Homotopy theory

4.1. **Homotopy.**

Definition 4.1. Let $f, g: X \to Y$ be two continuous functions between topological spaces. We say that f is *homotopic* to g if there exists a continuous function

$$H: [0,1] \times X \to Y$$

satisfying $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$.

Note 4.1. We denote by $H_s: X \to Y$ the function $H_s(x) = H(s, x)$. Note that each H_s is continuous and that $H_0 = f$ and $H_1 = g$.

Note 4.2. If f is homotopic to g we write $f \simeq g$

Proposition 4.2. *Homotopy is an equivalence relation on continuous functions from* X *to* Y.

4.2. **Path homotopy.**

Definition 4.3. Let X be a topological space and x and y be points in X. Then a path in X from x to y is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Note 4.3. If $x = y$ then we call the path a *loop* in X at x.

Definition 4.4. Two paths γ, γ' are called path homotopic if we have a continuous map $H\colon [0,1]\times[0,1] \to X$ such that, if we define $F_s(t) = F(s,t)$, then each $F_s\colon [0,1] \to X$ is a path from x to y and $F_0 = \gamma$ and $F_1 = \gamma'$.

Proposition 4.5. *Path homotopy is an equivalence relation on the set of all paths from x to y.*

Note 4.4. We denote the equivalence class of a path, or loop γ by $[\gamma]$.

Note 4.5. Notice that if we have a homotopy between two loops at x then each F_s is also a loop at x for every s. The set of all equivalence classes of loops at x is denoted $\pi_1(X,x)$ and called the *fundamental group* of X (at x).

Definition 4.6. If α and β are paths in X we call them *composable* if $\alpha(1) = \beta(0)$.

Given γ and β composable paths consider the function from $[0,1]$ to X defined by

$$\alpha \star \beta(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1. \end{cases}$$

By the Pasting Lemma this is a path from $\alpha(0)$ to $\beta(1)$ called the *product* of α and β. We call $\alpha \star \beta$ the *path product* of α and β.

Lemma 4.7. *If α and β are as above and α is homotopic to α' and β to β' then $\alpha \star \beta$ is homotopic to $\alpha' \star \beta'$.*

This lemma shows that there is a well-defined product of homotopy classes of paths and loops defined by $[\alpha][\beta] = [\alpha \star \beta]$. Denote by $\Pi(X)$ the set of all paths in X and define $s, t\colon \Pi(X) \to X$ by $s([\gamma]) = \gamma(0)$ and $t([\gamma]) = \gamma(1)$.

Proposition 4.8. *The pair $\Pi(X)$ and X define the morphisms and objects of a groupoid with the path product. The inverse of $[\gamma]$ is $[\gamma{-}1]$ where $\gamma^{-1}(t) = \gamma(1-t)$. The identity at $x \in X$ is the equivalence class of the constant path $e_x(t) = x$.*

This groupoid is called the homotopy groupoid of X and denoted $\Pi(X)$.

Definition 4.9. We say that a topological space X is path-connected if for any $x, y \in X$ there is a path from x to y.

Proposition 4.10. *The relation 'there is a path joining x to y' is an equivalence relation on any topological space.*

Note 4.6. The equivalence classes under this relation are called the *path-components* of X.

Proposition 4.11. *A topological space X is path-connected if and only if the homotopy groupoid is transitive.*

Note 4.7. The group of all morphisms beginning and ending at x in $\Pi(X)$ is denoted $\pi_1(X,x)$ and called the *fundamental group* of X. It is the set of all path-homotopy classes of loops as x with the path product.

We can apply the results from groupoids as follows: If α is a path from x to y and γ is a loop at x then we can define a loop at y by $(\alpha^{-1}\gamma)\alpha$ where $\alpha^{-1}(t) = \alpha(1-t)$. We define a map

$$I_{[\alpha]}\colon \pi_1(X,x) \to \pi_1(X,y)$$

by $I_{[\alpha]}([\gamma]) = [(\alpha^{-1}\gamma)\alpha] = [\alpha^{-1}(\gamma\alpha)]$. and it follows as before that:

Proposition 4.12. *The map $I_{[\alpha]}$ is a group isomorphism and $(I_{[\alpha]})^{-1} = I_{[\alpha^{-1}]}$.*

Note 4.8. Because of this proposition we often drop the reference to the point x for a path connected space X and just refer to the fundamental group $\pi_1(X)$ of X.

4.3. **Contractible maps.**

Definition 4.13. If the identity map on X is homotopic to a constant map then we call X *contractible*.

Example 4.1. If $X = Y = \mathbb{R}^n$ then the identity map is constractible to the constant map to zero by $F(s,x) = sx$.

Example 4.2. Let X be a star shaped region in $\mathbb{R}^n$, that is a region $X \subset \mathbb{R}^n$ with a point $x \in X$ with the property that that for every other point $y \in X$ the line segment from X to y is also in X. Then X is contractible.

Definition 4.14. A topological space is called *simply connected* if it is path connected and its fundamental group is zero.

Proposition 4.15. *A contractible space is simply-connected.*

Note 4.9. The converse is not true. We shall see later that $\pi_1(S^2) = 0$ and S^2 is certainly path-connected but it is not contractible.

If $f\colon X \to Y$ is a continous map then we can define a map $\pi_1(f) = f_*$ from $\pi_1(X,x)$ to $\pi_1(Y, f(x))$ as follows. Let γ be a loop at x then $f \circ \gamma$ is a loop at $f(x)$. It is easy to check that if γ is homotopic to γ' then $f \circ \gamma$ is homotopic to $f \circ \gamma'$ and so we can define $f_*([\gamma]) = [f \circ \gamma]$. It is also easy to check that f_* is a group homomorphism.

Let f and g be homotopic continuous maps from a space X to a space Y. Let $F\colon [0,1]\times X \to Y$ be a homotopy. Then we have $f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ and $g_*\colon \pi_1(X,x) \to \pi_1(Y, g(x))$. Define $\alpha(t) = F(t,x)$, then α is a path from $f(x)$ to $g(x)$. Recall the definition of $I_{[\alpha]}\colon \pi_1(Y, f(x)) \to \pi_1(Y, g(x))$ from above. Then we have

Proposition 4.16. *With the notation as in the preceeding discussion we have*

$$I_{[\alpha]} \circ f_* = g_*.$$

Definition 4.17. A map $f\colon X \to Y$ is called a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Definition 4.18. Two spaces are called *homotopy equivalent* if there is a homotopy equivalence between them.

Corollary 4.19. *If two space are homotopy equivalent then they have isomorphic fundamental groups.*

Example 4.3. A contractible space is homotopic to a point, that is to a space with only one element.

Example 4.4. The space $\mathbb{R}^2 - \{0\}$ is homotopy equivalent to S^1.

4.4. **The fundamental group of the circle.** We define a continuous map $p\colon \mathbb{R} \to S^1$ by $p(x) = (\cos(x), \sin(x))$. It follows from elementary calculus that we can cover S^1 by open sets U_i such that there are continuous maps $s_i\colon U_i \to \mathbb{R}$ such that $p(s_i(y)) = y$ for all $y \in S^1$. We have two important results.

Proposition 4.20 (Path lifting property.). *Let $y \in S^1$ and $x \in \mathbb{R}$ with $p(x) = y$. Let γ be a loop at x then there is a unique continuous map $\hat{\gamma}\colon [0,1] \to \mathbb{R}$ such that $p \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = x$.*

Note 4.10. We call $\hat{\gamma}$ a lift of γ or we say that it covers γ.

Proposition 4.21 (Covering homotopy property.). *Let $y \in S^1$ and $x \in \mathbb{R}$ with $p(x) = y$. Let γ and $\mathbb{R}ho$ be loops at x and $F\colon [0,1]\times[0,1] \to S^1$ be a homotopy from γ to $\mathbb{R}ho$. Let $\hat{\gamma}$ be a lift of γ with $\gamma(0) = x$. Then there is a unique lift of F to a map $\hat{F}\colon [0,1]\times[0,1] \to \mathbb{R}$ such that $p \circ \hat{F} = F$ and $F(0,t) = \hat{\gamma}(t)$ for all t.*

It follows that if γ is a loop in S^1 then $(1/2\pi)(\hat{\gamma}(0) - \hat{\gamma}(1))$ is an integer that depends only on the homotopy class of γ. We have

Proposition 4.22. *The map*

$$[\gamma] \mapsto \frac{1}{2\pi}(\hat{\gamma}(0) - \hat{\gamma}(1))$$

defines an isomorphism between $\pi_1(S^1, x)$ and $\mathbb{Z}$.

Note 4.11. We call this integer the degree or winding number of γ.

Note 4.12. If the loops in question are differentiable we can construct the winding number by the integral

$$\frac{1}{2\pi}\int \frac{1}{\gamma}\frac{d\gamma}{dt}dt.$$

Corollary 4.23. *The fundamental group of $\mathbb{R}^2 - \{0\}$ is $\mathbb{Z}$.*

Theorem 4.24 (Brouwer fixed point theorem). *If $f\colon D \to D$ is a continous map of the disk $D = \{x \in \mathbb{R}^2 \mid |x|^2 \le 1\}$ to itself then f has a fixed point, that is there is an $x \in D$ with $f(x) = x$.*

4.5. Fundamental group of a product.

Proposition 4.25. *Let $F: X \to Y \times Z$ be a function between topological spaces. Then it defines functions $f: X \to Y$ and $g: X \to Z$ by $F(x) = (f(x), g(x))$. Moreover any pair of functions f and g defines a function F in this manner. In such a situation F is continuous if and only if the two functions f and g are continuous.*

With this result it easy to prove:

Proposition 4.26. *The fundamental group of a product $X \times Y$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Example 4.5. The fundamental group of a torus $S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

4.6. **Van Kampen theorem.** If G and H are two groups we define the free product $G * H$ to consist of all finite 'words'

$$g_1 h_1 g_2 h_2 \dots g_k h_k$$

subject to the obvious identifications if some of the g_i or h_i are the identity. For example if e_H is the identity in H then $g_1 e_H g_2 h_2 = (g_1 g_2) h_2$. We define a product on $G * H$ by juxtaposing words and simplifying if necessary. For example if we justapose gh and $e_G h^{-1} g'$ the result would be $(gh)(e_G h^{-1} g') = gg'$.

If $S \subset G$ is a subset of a group define $\langle S \rangle$ to be the normal subgroup generated by S that is the smallest normal subgroup containing S.

Let $X = U \cup V$ where U and V are open and $U \cap V$ is **path-connected.** Define ι_U and ι_V to be the inclusion maps from $U \cap V$ into U and V respectively. Then we have

Theorem 4.27 (Van Kampen theorem (not proved))**.** *In the situation above the homotopy group of X is*

$$\pi_1(X) = \frac{\pi_1(U) * \pi_1(V)}{\langle \{\iota_U([\gamma^{-1}])\iota_V([\gamma]) \mid [\gamma] \in \pi_1(U \cap V)\} \rangle}.$$

Corollary 4.28 (Weak Van Kampen theorem)**.** *If $\pi_1(U) = \pi_1(V) = 0$ and $U \cap V$ is path connected then $\pi_1(U \cup V) = 0$.*

Corollary 4.29. *If $X = U \cup V$ and $U \cap V$ is simply connected then $\pi_1(X) = \pi_1(U) * \pi_1(V)$.*

Proposition 4.30. *The fundamental group of the n sphere for $n > 1$ is 0.*

Proposition 4.31. *The fundamental group of the plane with r points removed is the free group on r generators that is the free product of r copies of $\mathbb{Z}$.*

4.7. **Final thoughts about homotopy theory.** In this section I discussed some other aspects of homotopy theory that aren't for examination. I talked about the higher homotopy groups $\pi_n(X)$, and why they are abelian. Discussed suspension and how it shows that $\pi_k(S^n)$ is $\mathbb{Z}$ if $k = n$ and zero if $0 < k < n$. Drew the Hopf fibration. Also explained the Poincare conjecture. Also talked about the fundamental group of a Riemann surface and showed how to cut it up to make a polygon.

5. Homology of geometric complexes

5.1. Geometric complexes and polyhedra.

Definition 5.1. A set of $k + 1$ points $a_0, \dots, a_{k+1}$ in $\mathbb{R}^n$ is called *geometrically independent* if they lie in no $k - 1$ dimensional hyperplane.

Definition 5.2. Let $a_0, \dots, a_k$ be a geometrically independent set of points in $\mathbb{R}^n$. The k dimensional geometric simplex or k-simplex spanned by them is

$$< a_0 \dots a_k >= \left\{ \sum_{i=0}^{k} \lambda_i a_i \mid \sum_{i=0}^{k} \lambda_i = 1, \quad 0 \le \lambda_i \le 1 \right\}.$$

Note 5.1. The numbers $\lambda_0, \dots, \lambda_k$ are called the *barycentric co-ordinates* of $x = \sum_{i=0}^{k} \lambda_i a_i$. The subset of $< a_0 \dots a_k >$ consisting of all points with positive barycentric co-ordinates is called the *open k-simplex.*

Note 5.2. The a_i are called the *vertices* of the k-simplex $< a_0 \dots a_k >$.

Definition 5.3. A simplex σ_k is called a *face* of a simplex σ_n if every vertex of σ_k is a vertex of σ_n.

Definition 5.4. Two simplice σ_n and σ_m are called *properly joined* if $\sigma_n \cap \sigma_m = \emptyset$ or $\sigma_n \cap \sigma_m$ is a face of each of σ_m and σ_n.

Definition 5.5. A geometric complex (or *simplicial complex*) is a finite family K of geometric simplices which are properly joined and such that any face of a simplex in K is also in K.

The union of all the simplices in K with the subspace topology from $\mathbb{R}^n$ is denoted $|K|$ and called the *geometric carrier* of K or the polyhedron associated to K.

Definition 5.6. Let X be a topological space. If X is homeomorphic to $|K|$ then X is called *triangulable* and K is called a *triangulization* of K.

Definition 5.7. The closure of a simplex $\mathrm{Cl}\,\sigma_k$ is the complex consisting of σ^k and all its faces.

Definition 5.8. If K is a complex the r-skeleton is the set of all simplices of dimension less than or equal to k.

5.2. **Orientation of geometric complexes.**

Definition 5.9. An orientation of a complex is a choice of ordering up to an even permutation.

Note 5.3. We extend the notation introduced so that $< a_0 \dots a_k >= + < a_0 \dots a_k >$ denotes the oriented simplex with orientation determined by the ordering $a_0 \dots \alpha_k$ and $- < a_0 \dots a_k >$ denotes the simplex with the opposite orientation.

Definition 5.10. A complex K is called oriented if each simplex in K is oriented.

Definition 5.11. Let K be an oriented complex. Let σ^{p+1} and σ^p be two simplices in K of dimension $p+1$ and p respectively. Define $[\sigma^{p+1}, \sigma^p]$ as follows. If σ^p is not a face of σ^{p+1} then $[\sigma^{p+1}, \sigma^p] = 0$. If $\sigma^p =< a_0 \dots a_p >$ and v is the additional vertex in σ^{p+1} then $\sigma^{p+1} = [\sigma^{p+1}, \sigma^p] < v a_0 \dots a_p >$.

Theorem 5.12. *Let σ^p, σ^{p-2} be a p simplex and a $p-2$ simplex in an oriented complex K. Then*

$$\sum_{\sigma^{p-1} \in K} [\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0.$$

5.3. **Chains, cycles, boundaries and homology groups.**

Definition 5.13. Let K be an oriented complex. A p-chain is a formal linear finite linear combination $\sum m_i \sigma_i^p$ where the m_i are integers and the σ_i^p are from K. We denote by $C_p(K)$ the set of all p-chains.

Definition 5.14. We define the *boundary map* $\partial \colon C_p(K) \to C_{p-1}(K)$ by

$$\partial(\sum g_i \sigma_i^p) = \sum_{\sigma^{p-1}} g_i [\sigma_i^p, \sigma^{p-1}] \sigma^{p-1}.$$

Theorem 5.15.

$$\partial^2 = 0$$

Definition 5.16. A p-cycle is a p-chain in the kernel of ∂ which we denote by $Z_p(K)$. A p-boundary is an element of the image of ∂ which we denote by $B_p(K)$. We say two cycles are *homologous* if they differ by a boundary. We define $H_p(K) = Z_p(K)/B_p(K)$ to be the pth homology group of K.

Theorem 5.17. *The homology groups are independent of the choice of orientation on K*

Definition 5.18. Two simplices σ_0 and σ_1 in a complex K are said to be combinatorially connected if there exist verticest $a_0, \dots, a_p$ in K with a_0 a vertex of σ_0, a_p a vertex of σ_1 and such that $< a_0, a_1 >, < a_1, a_2 >, \dots, < a_{p-1}, a_p >$ are all in K.

Note 5.4. Combinatorial connectedness is an equivalence relation and the equivalence classes are called the cobinatorial components of K.

Proposition 5.19. *A complex K is combinatorially connected if and only if $|K|$ is path connected.*

Theorem 5.20. *If K is a complex then $H^0(K) = \mathbb{Z}^d$ where d is the number of combinatorial components of K.*

5.4. **Final comments.** The last result shows that $H^0(K)$ depends only on the topology of $|K|$. Much more is true, if K and L are complexes and $|K|$ and $|L|$ are homeomorphic then $H^p(K) = H^p(L)$ for all p.