Algebraic Topology IV 2004

Notes on Differential Forms – Michael Murray

1. The exterior algebra of a vector space.

If V is a finite dimensional vector space we define a k-linear map to be a map

$$\omega\colon V \times \cdots \times V \to \mathbb{R},$$

where there are k copies of V, which is linear in each factor. That is

$$\begin{aligned}\omega(v_1, \dots, v_{i-1}, \alpha v + \beta w, v_{i+1}, v_k) &= \alpha\omega(v_1, \dots, v_{i-1}, v, v_{i+1}, v_k)\\ &\quad + \beta\omega(v_1, \dots, v_{i-1}, w, v_{i+1}, v_k).\end{aligned}$$

We define a k-linear map ω to be totally antisymmetric if

$$\omega(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$$

for all vectors $v_1, \dots, v_k$ and all i. Note that it follows that

$$\omega(v_1, \dots, v, v, \dots, v_k) = 0$$

and if $\pi \in S_k$ is a permutation of k letters then

$$\omega(v_1, v_2, \dots, v_k) = \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation π. We denote the vector space of all k-linear, totally antisymmetric maps by $\Lambda^k(V^*)$. and call them k forms. If $k = 1$ the $\Lambda^1(V^*)$ is just V^* the space of all linear functions on V and if $k = 0$ we make the convention that $\Lambda^0(V^*) = \mathbb{R}$. We need to collect some results on the linear algebra of these spaces.

Assume that V has dimension n and that $v_1, \dots, v_n$ is a basis of V. Let ω be a k form. Then if $w_1, \dots, w_k$ are arbitrary vectors and we expand them in the basis as

$$w_i = \sum_{j=1}^{n} w_{ij} v_j.$$

then we have

$$\omega(w_1, \dots, w_k) = \sum_{j_1, \dots, j_k = 1}^{n} w_{1j_1} w_{2j_2} \dots w_{kj_k} \omega(v_{j_1}, \dots, v_{j_k})$$

so that it follows that ω is completely determined by its values on basis vectors. In particular if $k > n$ then $\Lambda^k(V^*) = 0$.

If α^1 and α^2 are two linear maps in V^* then we define an element $\alpha^1 \wedge \alpha^2$, called the wedge product of α^1 and α^2, in $\Lambda^2(V^*)$ by

$$\alpha^1 \wedge \alpha^2(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1).$$

More generally if $\omega \in \Lambda^p(V^*)$ and $\rho \in \Lambda^q(V^*)$ we define $\omega \wedge \rho \in \Lambda^{p+q}(V^*)$ by

$$\begin{aligned}&(\omega \wedge \rho)(w_1, \dots, w_{p+q})\\ &\qquad = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \operatorname{sgn}(\pi)\omega(w_{\pi(1)}, \dots, w_{\pi(p)})\rho(w_{\pi(p+1)}, \dots, w_{\pi(p+q)}).\end{aligned}$$

Assume that $\dim(V) = n$. Then we leave as an exercise the following proposition.

Proposition 1.1. *The direct sum*

$$\Lambda(V^*) = \bigoplus_{k=1}^{n} \Lambda^k(V^*)$$

with the wedge product is an associative algebra.

We call $\Lambda(V^*)$ the exterior algebra of V^*. We call an element $\omega \in \Lambda^k(V^*)$ an element of degree k. Because of associativity we can repeatedly wedge and disregard brackets. In particular we can define the wedge product of m elements in V^* and we leave it as an exercise to show that

$$\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^m = \sum_{\pi \in S_m} \operatorname{sgn}(\pi)\alpha^1(v_{\pi(1)})\alpha^2(v_{\pi(2)})\ldots\alpha^m(v_{\pi(m)}).$$

Notice that

$$\alpha^1 \wedge \cdots \wedge \alpha^i \wedge \alpha^{i+1} \wedge \cdots \wedge \alpha^m = -\alpha^1 \wedge \cdots \wedge \alpha^{i+1} \wedge \alpha^i \wedge \cdots \wedge \alpha^m$$

and that

$$\alpha^1 \wedge \cdots \wedge \alpha \wedge \alpha \wedge \cdots \wedge \alpha^m = 0.$$

Still assuming that V is n dimensional choose a basis $v_1, \ldots, v_n$ of V. Define the dual basis of V^*, $\alpha^1, \ldots, \alpha^n$, by

$$\alpha^i(v_j) = \delta^i_j$$

for all i and j. We want to define a basis of $\Lambda^k(V^*)$. Define elements of $\Lambda^k(V)$ by choosing k numbers $i_1, \ldots, i_k$ between 1 and n and considering

$$\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

We have

Proposition 1.2. *The vectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$ are a basis for $\Lambda^k(V^*)$.*

It is sometimes useful to sum over all k-tuples $i_1, \ldots, i_k$ not just ordered ones. We can do this — an keep the uniqueness of the coefficients $\omega_{i_1\ldots i_k}$ — if we demand that they be antisymmetric. That is

$$\omega_{j_1\ldots j_i j_{i+1}\ldots j_k} = -\omega_{j_1\ldots j_{i+1} j_i \ldots j_k}.$$

Then we have

$$\begin{aligned}\omega &= \textstyle\sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1\ldots i_k}\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}\\ &= \textstyle\sum_{1 \leq i_1, \ldots, i_k \leq n} \frac{1}{k!}\omega_{i_1\ldots i_k}\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.\end{aligned}$$

We will need one last piece of linear algebra called *contraction.* Let $\omega \in \Lambda^k(V)$ and $v \in V$. Then we define a $k-1$ form $\iota_v\omega$, the contraction of ω and v by

$$\iota_v(\omega)(v_1, \ldots, v_{k-1}) = \omega(v_1, \ldots, v_{k-1}, v)$$

where $v_1, \ldots, v_{k-1}$ are any $k-1$ elements of V.

Example 1.1. Consider the vector space $\mathbb{R}^3$. Then we know that zero forms and one forms are just real numbers and linear maps respectively. Notice that in the case of $\mathbb{R}^3$ we can identify any linear map v with the vector $v = (v^1, v^2, v^3)$ where

$$v(x) = \sum_{i=1}^{3} v^i x^i.$$

Let α^i be the basis of linear functions defined by $\alpha^i(x) = x^i$. We have seen that every two form ω on $\mathbb{R}^3$ has the form

$$\omega = \omega_1\alpha^2 \wedge \alpha^3 + \omega_2\alpha^3 \wedge \alpha^1 + \omega_3\alpha^1 \wedge \alpha^2.$$

Every three-form μ takes the form

$$\mu = a\alpha^1 \wedge \alpha^2 \wedge \alpha^3.$$

It follows that in $\mathbb{R}^3$ we can identify three-forms with real numbers by identifying μ with a and we can identify two-forms with vectors by identifying ω with $(\omega_1, \omega_2, \omega_3)$.

It is easy to check that with these identifications the wedge product of two vectors v and w is identified with the vector $v \times w$. In other words wedge product corresponds to cross product.

2. Differential forms.

We can now apply the constructions of the previous section to the tangent space to a manifold. We define a k-form on the tangent space at $x \in M$ to be an element of

$$\Lambda^k T_x^* M.$$

We want to define k-form 'fields' in the same way we define vector fields except that we do not call them k-form fields we call them differentiable k-forms or sometimes just k-forms. Choose co-ordinates (U, ψ) on M. Then $\omega(x)$ in $\Lambda^k(T_x^* M)$ can be be written as

$$\omega(x) = \sum_{i_1,\dots,i_k} \frac{1}{k!} \omega_{i_1,\dots,i_k} d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}$$

at each $x \in U$. Hence we have defined a function

$$\omega_{i_1,\dots,i_k} \colon U \to \mathbb{R}$$

for each set of k indices. We call these functions the *components* of ω with respect to the co-ordinate chart. The components satisfy the anti-symmetry conditions in the previous section. We can also define the $\omega_{i_1,\dots,i_k}$ as

$$\omega_{i_1,\dots,i_k} = \omega(\frac{\partial}{\partial \psi^{i_1}}, \dots, \frac{\partial}{\partial \psi^{i_k}}).$$

We define a smooth differential form by

Definition 2.1 (Differential form.)**.** A differential form ω is smooth if its components with respect to a collection of co-ordinate charts whose domains cover M are smooth.

We denote by $\Omega^k(M)$ the set of all smooth differentiable k forms on M. Notice that $\Omega^0(M)$ is just $C^\infty(M)$ the space of all smooth functions on M.

Using the equation

$$\omega_{i_1,\dots,i_k} = \omega(\frac{\partial}{\partial \psi^{i_1}}, \dots, \frac{\partial}{\partial \psi^{i_k}}).$$

for the components of the differential form we can calculate the way the components change if we use another co-ordinate chart (V, χ). We have

$$\frac{\partial}{\partial \psi^i} = \sum_{a=1}^n \frac{\partial \chi^a}{\partial \psi^i} \frac{\partial}{\partial \chi^a}$$

and substituting this into the formula gives

$$\omega_{i_1,\dots,i_k} = \sum_{a_1,\dots,a_k=1}^n (\frac{\partial \chi^{a_1}}{\partial \psi^{i_1}} \cdots \frac{\partial \chi^{a_n}}{\partial \psi^{i_n}}) \omega_{a_1,\dots,a_k}.$$

2.1. Exterior derivative.

The usual derivative on functions defines a linear differential operator

$$d \colon \Omega^0(M) \to \Omega^1(M).$$

As well as being linear d satisfies the Leibniz rule:

$$d(fg) = f dg + (df) g.$$

We have

Proposition 2.2. *If the dimension of M is n then there are unique linear maps*

$$d \colon \Omega^p(M) \to \Omega^{p+1}(M)$$

for all $p = 0, \dots, n-1$ satisfying:

1. *If $p = 0$ d is the usual derivative,*
2. *$d^2 = 0$, and*
3. *$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^p \omega \wedge (d\rho)$ where $\omega \in \Omega^p(M)$ and $\rho \in \Omega^q(M)$.*

This map d is called the *exterior derivatice.*

Example 2.1. Recall from 1.1 the way in which we identified one-forms and two-forms on $\mathbb{R}^3$ with vectors. It follows that differentiable one and two forms on $\mathbb{R}^3$ can be identified with vector-fields. Similarly zero and three forms are functions. With these identifications it is straightforward to check that the exterior derivative of zero, one and two forms corresponds to the classical differential operators grad, curl and div.

If (U, ψ) is a co-ordinate chart then the proposition shows us how to define the exterior derivative of a differential form locally. If we have

$$\omega(x) = \sum_{i_1,\dots,i_k} \frac{1}{k!}\omega_{i_1,\dots,i_k} d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}$$

then

$$d\omega(x) = \sum_{j,i_1,\dots,i_k} \frac{1}{k!}\frac{\partial \omega_{i_1,\dots,i_k}}{\partial \psi^j} d\psi^j \wedge d\psi^{i_1} \wedge \cdots \wedge d\psi^{i_k}.$$

2.2. **Pulling back differential forms.** We have seen that if $f\colon M \to N$ is a smooth map then it has a derivative or tangent map $T_x(f)$ that acts on tangent vectors in T_xM by sending them to $T_{f(x)}N$. Moreover $T_x(f)$ is linear. Recall that if $X\colon V \to W$ is a linear map between vector spaces then it has an adjoint or dual $X^*\colon W^* \to V^*$ defined by

$$X^*(\xi)(v) = \xi(X(v))$$

where $\xi \in W^*$ and $v \in V$. Notice that X^* goes in the opposite direction to X. So we have a linear map called the cotangent map

$$T_x^*(f)\colon T_{f(x)}N \to T_xM$$

which is just the adjoint of the tangent map. It is defined by

$$T_x^*(f)(\omega)(X) = \omega(T_x(f)(X).$$

This action defines a map on differential forms called the pull-back by f and denoted f^*. if $\omega \in \Omega^k(N)$ then we define $f^*(\omega) \in \Omega^k(M)$ by

$$f^*(\omega)(x)(X_1, \dots, X_k) = \omega(f(x))(T_x(f)(X_1), \dots, T_x(f)(X_k))$$

for any $X_1, \dots, X_k$ in T_xM.

Notice that if ϕ is a zero form or function on N then $f^{-1}(\phi) = \phi \circ f$. The pull back map

$$f^*\colon \Omega^q(N) \to \Omega^q(M).$$

satisfies the following proposition.

Proposition 2.3. *If $f\colon M \to N$ is a smooth map and ω and μ is a differential form on N then:*

1. $df^*(\omega) = f^*(d\omega)$*, and*
2. $f^*(\omega \wedge \mu) = f^*(\omega) \wedge f^*(\mu)$*.*

3. Integration of differential forms

Let $U \subset \mathbb{R}^n$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then it is well known that if $f\colon \psi(U) \to \mathbb{R}$ is a function then

$$\int_U f \circ \psi |\det\left(\frac{\partial \psi^i}{\partial x^j}\right)| dx^1 \dots dx^n = \int_{\psi(U)} f dx^1 \dots dx^n.$$

In this formula we regard $dx^1 \dots dx^n$ as the symbol for Lebesgue measure. However it is very suggestive of the notation for differential forms developed in the previous section.

If ω is a differential n form on $V = \psi(U)$ then we can write it as

$$\omega(x) = f(x)dx^1 \wedge \cdots \wedge dx^n.$$

If we pull it back with the diffeomorphism ϕ then, as we seen before,

$$\phi^*(\omega) = f(x)\det\left(\frac{\partial \psi^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n.$$

So differential n forms transform by the determinant of the jacobian of the diffeomorphism and Lebesgue measure transforms by the absolute value of the determinant of the jacobian of the diffeomorphism. We define the integral of a differential n form by

$$\int_V \omega = \int_V f(x) dx^1 \dots dx^n$$

when $\omega = f(x) dx^1 \wedge \dots \wedge dx^n$. Alternatively we can write this as

$$\int_V \omega = \int_V \omega(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) dx^1 \dots dx^n.$$

Call a diffeomorphism $\psi \colon U \to V$ *orientation preserving* if

$$\det \left(\frac{\partial \psi^i}{\partial x^j} \right) (x) > 0$$

for all $x \in U$. Then we have

Proposition 3.1. *If $\psi \colon U \to \psi(U)$ is an orientation preserving diffeomorphism and ω is a differential n form on $\psi(U)$ then*

$$\int_{\psi(U)} \omega = \int_U \psi^*(\omega).$$

3.1. **Orientation.** Let V be a real vector space of dimension n. Then define $\det(V) = \Lambda^n(V)$. This is a real, one dimensional vector space. So the set $\det(V) = \{0\}$ is *disconnected*. An orientation of the vector space V is a choice of one of these connected components. If X is an invertible linear map from V to V then it induces a linear map from $\det(V) \to \det(V)$ which is therefore multiplication by a complex number. This number is just $\det(X)$ the determinant of X If M is a manifold of dimension n then the same applies to M; $\det(T_x M) - \{0\}$ is a disconnected set. We define

Definition 3.2. A manifold is orientable if there is a non-vanishing n-form on M. Otherwise it is called non-orientable.

If η and ζ are two non-vanishing n forms then $\eta = f\zeta$ for some function f which is either strictly negative or strictly positive. Hence the set of non-vanishing n forms divides into two sets. We have

Definition 3.3 (Orientation)**.** An orientation on M is a choice of one of these two sets.

An orientation defines an orientation on each tangent space $T_x M$. We call an n form positive if it coincides with the chosen orientation negative otherwise. We say a chart (U, ψ) is positive or oriented if $d\psi^1 \wedge \dots \wedge d\psi^n$ is positive. Note that if a chart is not positive we can make it so by changing the sign of one co-ordinate function so oriented charts exits. If we chose two oriented charts then we have that

$$\chi \circ \psi^{-1}_{|\psi(U \cap V)}$$

is an oriented diffeomorphism. The converse is also true.

We can use this proposition to define the integral of differential forms on a manifold. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ be a covering of M by oriented co-ordinate charts. Choose a partition of unity ϕ_α subordinate to U_α. Then if ω is a differential n form we can write

$$\omega = \sum_\alpha \phi_\alpha \omega$$

where the support of $\phi_a \omega$ is in U_α. First we define the integral of each of the forms $\phi_\alpha \omega$

$$\int_M \phi_a \omega = \int_{\psi_\alpha(U_\alpha)} (\psi^{-1})^*(\phi_a \omega).$$

Then we define the integral of ω to be

$$\int_M \omega = \sum_{\alpha \in I} \int_{\psi_\alpha(U_\alpha)} (\psi^{-1})^*(\omega).$$

This is independent of all the choices we have made except the choice of orientation.

Typically we don't calculate an integral in this way. Instead if M is an n dimensional manifold and ω is an n-form then we look for an oriented co-ordinate chart (U, ψ) that covers all of M except a set of measure zero. Then

$$\int_M \omega = \int_U \omega = \int_{\psi(U)} \omega_{12\ldots n} d\psi^1 \ldots d\psi^n.$$

4. Stokes theorem.

Recall the Fundamental Theorem of Calculus: If f is a differentiable function then

$$f(b) - f(a) = \int_b^a \frac{df}{dt}(x) dx.$$

In the language we have developed in the previous section this can be written as

$$f(b) - f(a) = \int_{[a,b]} df$$

where we orient the 1-dimensional manifold $[a, b]$ in the positive direction. Stokes theorem is a generalisation of this and Stokes theorem, Green's theorem, Gauss' theorem and the Divergence theorem.

Theorem 4.1 (Stoke's theorem)**.** *Let M be an oriented manifold with boundary of dimension n and let ω be a differential form of degree $n - 1$ with compact support then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Appendix A. Partitions of unity.

If M is a manifold a partition of unity is a collection of smooth non-negative functions $\{\rho_\alpha\}_{\alpha \in I}$ such that every $x \in M$ has neighbourhood on which only a finite number of the ρ are non-vanishing and such that $\sum_{\alpha \in I} \rho_\alpha = 1$.

Recall that if $f : M \to \mathbb{R}$ is smooth function then we define $\text{supp}(f)$ to be the closure of the set on which f is non-zero. There are two basic existence results on a paracompact, Hausdorff manifold.

(1) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M there is a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ with $\text{supp}(\rho_\alpha) \in U_\alpha$. Such a partition of unity is called subordinate to the cover.
(2) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M there is a partition of unity $\{\rho_\alpha\}_{\beta \in J}$, with a possibly different indexing set J with each $\text{supp}(\rho_\beta)$ in some U_α.