Algebraic Topology IV 2004

Assignment 1 – Due 14th September

1. Consider the set $X = \{a, b, c\}$. Prove that $\mathcal{P}(X)$ has 256 subsets. Find four topologies on X which are different in the sense that they are not equal and one cannot be obtained from the other by permuting the elements of X. Find two subsets of $\mathcal{P}(X)$ which are not topologies. If you follow the links from `http://mathworld.wolfram.com/Topology.html` you will find a listing of the number of topologies on a finite set. For X it is 29 if you do allow permutations of elements.

2. (a) If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X show that $\mathcal{T}_1 = \mathcal{T}_2$ if and only if for every $x \in B_1 \in \mathcal{B}_1$ there is a $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$ and for every $x \in B_2 \in \mathcal{B}_2$ there is a $B_1 \in \mathcal{B}_1$ with $x \in B_1 \subseteq B_2$.

(b) Show that the following bases on $\mathbb{R}^2$ give rise to the same topology.

$$\mathcal{B}_1 = \{B(x, \delta) \mid x \in \mathbb{R}^2, \delta > 0\}$$
$$\mathcal{B}_2 = \{\{(y^1, y^2) \in \mathbb{R}^2 \mid \max\{|x^1 - y^1|, |x^2 - y^2|\} < \delta\} | (x_1, x_2) \in \mathbb{R}^2, \delta > 0\}.$$

I will accept a *careful* drawing as a justification for the inclusions you need to prove.

3. Let X, Y and Z be topological spaces. Show that $F\colon X \to Y \times Z$ defined by $F(x) = (f(x), g(x))$ for $f\colon X \to Y$ and $g\colon X \to Z$ is continuous if and only if f and g are continuous.

4. Let X be a topological space containing points x, y, z and w. Let α be a path from x to y, β a path from y to z and γ a path from z to w. Choose numbers $0 < a < b < 1$. Define a path $\alpha \star \beta \star \gamma$ from x to w by

$$\alpha \star \beta \star \gamma = \begin{cases} \alpha(t/a) & 0 < t < a \\ \beta((t-a)/(b-a)) & a < t < b \\ \gamma((t-b)/(1-b)) & b < t < 1 \end{cases}$$

Show that $\alpha \star \beta \star \gamma$ is a continuous path from x to z. Show that the homotopy class of $\alpha \star \beta \star \gamma$ is independent of a and b.

5. Let X be a topological space. If γ is a path from x to y in X denote by

$$I_\gamma\colon \pi_1(X, x) \to \pi_1(X, y)$$
$$[f] \mapsto [\gamma^{-1}][f][\gamma]$$

the group homomorphism discussed in lectures. If ρ is another path from x to y show that $I_\gamma = I_\rho$ if $\gamma \simeq_p \rho$.

6. Define real projective space, $\mathbb{R}P_k$, to be the set of all lines through the origin in $\mathbb{R}^{k+1}$. Define a map $p\colon S^k \to \mathbb{R}P_k$ from the sphere to real projective space by mapping a unit vector to the line containing it. It is possible to make $\mathbb{R}P_k$ into a topological space so that p is continuous and moreover the following Propositions hold:

Proposition (Path lifting property). *Let $y \in \mathbb{R}P_k$ and $x \in S^k$ with $p(x) = y$. Let γ be a loop at x then there is a unique continuous map $\hat{\gamma}\colon [0,1] \to S^k$ such that $p \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = x$.*

Proposition (Covering homotopy property). *Let $y \in \mathbb{R}P_k$ and $x \in S^k$ with $p(x) = y$. Let γ and ρ be loops at x and $F\colon [0,1] \times [0,1] \to \mathbb{R}P_k$ be a path-homotopy from γ to ρ. Let $\hat{\gamma}$ be a lift of γ with $\gamma(0) = x$. Then there is a unique lift of F to a map $\hat{F}\colon [0,1] \times [0,1] \to S^k$ such that $p \circ \hat{F} = F$ and $F(0,t) = \hat{\gamma}(t)$ for all t.*

Note that these are similar to the results used to calculate $\pi_1(S^1)$.

Show that $p(u) = p(v)$ if and only if $u = \pm v$. Show that if $\gamma(t) = (\cos(t\pi), \sin(t\pi), 0, \ldots, 0)$ then $p \circ \gamma$ defines a loop (i.e a path that begins and ends at the same point) in $\mathbb{R}P_k$. You may assume that p is continuous.

Calculate the fundamental group of $\mathbb{R}P_k$ for $k > 1$.