Spectral curves of non-integral hyperbolic monopoles

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1 Introduction

In this paper we shall study hyperbolic monopoles with gauge group SU_n. These are solutions of the Bogomolny equations

$$*F_a = \nabla_a(i\phi) \tag{1.1}$$

over hyperbolic 3-space $\mathbb{H}^3$ which satisfy certain boundary conditions, the most important of which is

$$|\phi| \to p > 0 \text{ at } \infty. \tag{1.2}$$

In (1.1), a is an SU_n-connection on the trivial bundle over $\mathbb{H}^3$, F_a is its curvature, $i\phi$ (the Higgs field) is a section of the adjoint bundle, and $*$ is the Hodge $*$-operator on $\mathbb{H}^3$. We regard two monopoles as the same if they are gauge-equivalent. (The reason for our apparently eccentric notation for the Higgs field will become clear in §3.) We shall develop the twistor description of solutions of (1.1) from a somewhat novel point of view and then impose the boundary conditions that permit the spectral data of the monopole to be defined. An interesting consequence of the twistor description is a statement that, roughly speaking, hyperbolic monopoles are determined by their boundary data: this is reminiscent of one of the main results of [8] about certain SU_2-monopoles, but the details are different.

The motivation for this work comes from various sources. First of all, in view of the richness of the theory of Euclidean monopoles (an introduction to which can be found in the book of Atiyah and Hitchin [3]) it is to be expected that there is a similarly rich theory in the hyperbolic case. Indeed one may hope that the Euclidean theory could be recovered from the hyperbolic one by letting the curvature of $\mathbb{H}^3$ go to zero.

The second reason to study hyperbolic monopoles in detail comes from statistical mechanics. It has been noticed by Atiyah and Murray [4, 2] that the family of algebraic curves that arise as higher genus solutions of the chiral Potts model [5, 6] (one of the exactly solvable models in statistical mechanics) should be spectral curves for 'hyperbolic monopoles with $p = 0$'. Now by combining (1.1) with the Bianchi identity $d_a F_a = 0$ one finds

$$\Delta|\phi|^2 = 2|\nabla_a \phi|^2 \geq 0$$

(cf. [17, IV.9–10]) and it follows from the maximum principle that $|\phi|$ is bounded by its asymptotic value p. So every monopole with $p = 0$ is *flat* in the sense that the connection a is flat and ϕ is constant, $\nabla_a \phi = 0$. [This same argument shows that any monopole on a *compact* 3-manifold has to be flat and motivates their study on open manifolds as well as the condition $p > 0$ in (1.2).] Thus the limit $p \to 0$ is potentially very interesting and it is hoped that more information about it can be obtained from a more complete theory of hyperbolic monopoles. Note that by rescaling, this limit is equivalent to keeping p fixed while letting the curvature of hyperbolic space go to $-\infty$ so that this could be thought of as the opposite of the Euclidean limit.

Finally there is an interesting link with Hitchin's recent work [12] on $SO(3)$-invariant Einstein metrics. As pointed out in [12, §5] some of these live on spaces of spectral curves of hyperbolic monopoles and it would be of considerable interest to fit them into a full theory. (Note also that the present work supplies proofs of some of the assertions about hyperbolic monopoles that are made in that section of Hitchin's paper.)

The theory of hyperbolic monopoles began with the work of Atiyah [1] and was further advanced by Braam and Austin in [8]. However, in those papers only *integral* hyperbolic monopoles were considered, those for which the asymptotic eigenvalues of ϕ are integers. The reason for this is that these authors were really studying circle-invariant instantons on S^4 and using the conformal isometry $S^4 - S^2 \simeq S^1 \times \mathbb{H}^3$ (S^2 being the fixed set of the S^1-action) to identify such instantons with hyperbolic monopoles. Then the 'quantization' of ϕ_∞ comes exactly from the compactness of the circle. One of the novelties of our approach is the use of the projective model of $\mathbb{H}^3$ (recalled in §2) to identify hyperbolic monopoles with dilation-invariant solutions of the self-duality equations on the open cone of time-like vectors in Minkowski 4-space M. This is very natural (for example, the action of the isometry group $SO_0(1,3)$ of $\mathbb{H}^3$ is completely clear) and it is also very convenient since the twistor theory of Minkowski space is rather easy to work with. And since now we have an action of the non-compact group $\mathbb{R}_+$, quantization of the Higgs field is not forced. In §3, we shall derive what we shall call the *Hitchin–Ward correspondence* between holomorphic vector bundles on the twistor space Z of $\mathbb{H}^3$ and solutions of the Bogomolny equations (1.1). The main result of this section (Theorem 3.1) will not surprise anyone who is familiar with the monopole literature, but the proof we sketch involves some very useful new ideas which owe a lot to discussions with Toby Bailey and Michael Eastwood. [The application of these ideas to the Penrose transform will receive a systematic treatment in a forthcoming paper by Bailey, Eastwood and the second author.]

In §4 we find a very elegant way to describe the natural decay conditions for functions on $\mathbb{H}^3$ and use this to supplement (1.2) with conditions on the rate of decay of (∇_a, ϕ) which guarantee that the monopole determines, and is determined by, its spectral data. The construction of the spectral data is a generalization to the non-integral case of the construction of spectral curves in [1], and is the natural hyperbolic analogue of the constructions of [9, 11].

The paper ends with a short appendix which gathers some necessary results about first-order ordinary differential equations.

2 The projective model of hyperbolic space and its twistor space

One of the most natural descriptions of hyperbolic 3-space $\mathbb{H}^3$ is as the space of rays in the open cone U of future-pointing time-like vectors in Minkowski space M. Recall that M is a real 4-dimensional vector space equipped with an indefinite inner product $X.Y$ of signature $(+,-,-,-)$. Recall that $X \neq 0$ in M is called time-like if $X.X > 0$, null if $X.X = 0$ and space-like if $X.X < 0$.

In standard coordinates (X_0, X_1, X_2, X_3) relative to which

$$|X|^2 := X.X = X_0^2 - X_1^2 - X_2^2 - X_3^2$$

we have

$$U = \{X \in M : X_0 > 0 \text{ and } X_0^2 > X_1^2 + X_2^2 + X_3^2\}.$$

The multiplicative group $\mathbb{R}_+$ of positive real numbers acts by scalar multiplication on U and we set $U/\mathbb{R}_+ = \mathbb{H}^3$. We shall often write $[X]$ for the $\mathbb{R}_+$-orbit of X and π for the quotient map. The salient features of the relation between the geometry of U and the geometry of $\mathbb{H}^3$ are as follows.

- The Riemannian metric on $\mathbb{H}^3$ is given by $h = -\operatorname{Hess} \log |X|^2$ (thought of as a symmetric tensor on U).
- The geodesics of h correspond to the time-like 2-planes in U. (Recall that a 2-plane Π is *time-like* if it contains 2 distinct null vectors.) In other words, the geodesic joining $[X]$ to $[Y]$ is just given by the 2-plane spanned by X and Y.

- The geodesic distance between $[X]$ and $[Y]$ is given by

$$\mathrm{dist}([X],[Y]) = \cosh^{-1}\frac{X.Y}{|X||Y|}.$$

- The identity component $SO_0(1,3)$ of the group $SO(1,3)$ of linear transformations that preserve $|X|^2$ acts transitively and isometrically on $\mathbb{H}^3$. This is clear either from the definition of h or the formula for geodesic distance.

Note that by restricting h to a slice of the $\mathbb{R}_+$-action on U, we can obtain a more explicit model of $\mathbb{H}^3$. For example if one uses the coordinates

$$t = |X| > 0, X_1 + iX_2 = u$$

on the slice $X_0 - X_3 = 1$, we recover the upper half-space model of $\mathbb{H}^3$ with $h = t^{-2}(dt^2 + |du|^2)$.

The twistor space P of U is by definition the space of null geodesics γ in U [13, 19]. (A null geodesic in U is just a straight line with null tangent vector, that meets U.) Let $C \subset P \times U$ be the submanifold of pairs (γ, X) such that X lies on γ and let $\mu : C \to P$, $\nu : C \to U$ be the restrictions of the first and second projections. We call C the correspondence space. By definition, if γ is a point of P, then $\nu\mu^{-1}(\gamma)$ is a null geodesic in U; if X is a point of U, then $\mu\nu^{-1}(X)$ is an embedded 2-sphere in P. Now P sits naturally (i.e. $SO_0(1,3)$-invariantly) inside projective 3-space $\mathbb{CP}_3$ in such a way that a sphere in P is of the form $\mu\nu^{-1}(X)$ if and only if it is a complex projective line. We shall now explain this in more detail.

First some notation: for $w \in \mathbb{C}^2$, we put

$$w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, Jw = \begin{pmatrix} -w_1 & w_0 \end{pmatrix}, w^* = \begin{pmatrix} \bar{w}_0 & \bar{w}_1 \end{pmatrix}, \sigma w = (Jw)^* = \begin{pmatrix} -\bar{w}_1 \\ \bar{w}_0 \end{pmatrix}.$$

Thus σ is a quaternionic structure, $\sigma^2 = -1$. Let $\langle w, z\rangle = w^* z$. Now let homogeneous coordinates on $\mathbb{CP}_3$ be $[w, z]$ where w and z lie in $\mathbb{C}^2$ and let

$$P = \{[w, z] \in \mathbb{CP}_3 : \langle w, z\rangle > 0\};$$

this is an open subset of the real hypersurface $\{\mathrm{Im}\langle w, z\rangle = 0\} \subset \mathbb{CP}_3$. Identify M with the space of 2×2 complex Hermitian matrices:

$$(X_0, X_1, X_2, X_3) \mapsto X = \begin{pmatrix} X_0 + X_3 & X_1 + iX_2 \\ X_1 - iX_2 & X_0 - X_3 \end{pmatrix}, \ |X|^2 = \det X,$$

so that U becomes the cone of positive-definite matrices in M. Then $C \subset P \times U$ is given by the incidence relation

$$w = Xz. \tag{2.1}$$

From this it is clear, fixing X, that $\mu\nu^{-1}(X)$ is indeed a complex projective line, as mentioned above. On the other hand, solving for X gives the parametrized null geodesic

$$t \to \gamma_{[w,z]}(t) = ww^*/\langle w, z\rangle + t\sigma z\, \sigma z^*/|z|^2,\ t \in \mathbb{R}_+. \tag{2.2}$$

(Note that with this description of M: $V \neq 0$ is null iff V has rank 1 iff $V = vv^*$ for some $v \in \mathbb{C}^2$.) Although we have used coordinates to describe P, the natural symmetry is manifest. Recall that the action $X \mapsto gXg^*(g \in SL_2(\mathbb{C}))$ corresponds to the action of $SO(1,3)$ on M via the double-cover $SL_2(\mathbb{C}) \to SO(1,3)$. Then the induced action on P is given by $(w, z) \to (gw, (g^*)^{-1}z)$.

To find the twistor space for $\mathbb{H}^3$ we pass to the quotient by $\mathbb{R}_+$. Now the $\mathbb{R}_+$-action on U induces the action

$$\lambda(w, z) = (\lambda^{1/2}w, \lambda^{-1/2}z)$$

on P, so the quotient is the complex manifold

$$P/\mathbb{R}_+ = Z = \{([w], [z]) \in \mathbb{CP}_1 \times \mathbb{CP}_1 : \langle w, z\rangle \neq 0\}; \tag{2.3}$$

this is the twistor space of $\mathbb{H}^3$. A point of Z corresponds to the $\mathbb{R}_+$-orbit in U of the geodesic (2.2). Since this orbit is an oriented timelike 2-plane, we conclude that the points of Z parameterize the oriented geodesics in $\mathbb{H}^3$. Thus there is a natural involution of Z which switches the orientation of geodesics; this is given on P by the map

$$\sigma(w, z) = (\sigma z, \sigma w)$$

which induces the inversion $I : X \mapsto X/|X|^2$ of U. On the other hand, the $\mathbb{R}_+$-orbit of $X \in U$ evidently determines the curve

$$(Jw)Xz = 0 \tag{2.4}$$

on Z, a smooth curve in the linear system $|\mathcal{O}(1,1)|$ that is *real* in the sense of being σ-invariant. Since every real curve in $|\mathcal{O}(1,1)|$ has the form (2.4), such *real lines*, as we shall call them, are in 1:1-correspondence with the points of $\mathbb{H}^3$.

Remark 1. By restricting to slices of U, we can obtain formulae for this correspondence using the other models of $\mathbb{H}^3$. For example, if we take $X_0 - X_3 = 1$,

$$X = \begin{pmatrix} t^2 + |u|^2 & u \\ \bar{u} & 1 \end{pmatrix}$$

and (2.4) reduces to the equation

$$(z_1 + \bar{u}z_0)(w_0 - uw_1) = t^2 w_1 z_0$$

which gives the correspondence in upper half-space coordinates (u, t).

Remark 2. Some readers may be familiar with the approach to three-dimensional twistor theory given by Hitchin in [9]. There Hitchin shows that if M is a Riemannian three manifold the space Z of all oriented geodesics carries a natural almost complex structure which is integrable in the case that M has constant curvature. In the case of hyperbolic space it is easily seen, for example in the ball model that every geodesic is determined by its intersection with the sphere at infinity. An oriented geodesic is determined by the ordered pair of past endpoint and future endpoint. Every possible pair occurs except those corresponding to a geodesic beginning and ending at the same point. So as a manifold Z is $S^2 \times S^2$ with the diagonal deleted. To identify Z as a complex manifold we need to note that to obtain the natural complex structure of Hitchin it is better to parametrise an oriented geodesic by its future endpoint and the *antipode* of its past endpoint. Hence Z as a complex manifold is isomorphic to $S^2 \times S^2$ minus the anti-diagonal or precisely what we have shown in equation (2.3). We shall have more to say about this picture below in section 3.4.

3 Self-dual connections, hyperbolic monopoles and holomorphic bundles

In this section we shall explain the Ward–Hitchin transform between holomorphic vector bundles on Z and solutions of the Bogomolny equations. Before we can state the precise result we need some definitions. Let us agree to call a holomorphic vector bundle $\widetilde{E} \to Z$ *non-degenerate* if the restriction of $\widetilde{E}$ to any real line on Z is (holomorphically) trivial. Because we are starting with SU_n monopoles we expect the bundle on Z to be real in some appropriate sense. We define a *real* bundle to be one which has a real structure, that is an antiholomorphic map $\tilde{\sigma} : \widetilde{E} \to \widetilde{E}^*$ (which is antilinear on each fibre), that covers σ. Finally if we have a bundle $\widetilde{E} \to Z$ which is both non-degenerate and real the space of holomorphic sections over any real section has a natural hermitian inner product defined using the real structure. We define the bundle $\widetilde{E} \to Z$ to be *positive-definite* if this hermitian structure is positive definite. We then have

Theorem 3.1 *There is a natural bijection between:*

(i) isomorphism classes of non-degenerate holomorphic $SL_n(\mathbb{C})$-bundles $\widetilde{E} \to Z$, and

(ii) gauge equivalence classes of solutions of the $SL_n(\mathbb{C})$-Bogomolny equations on $\mathbb{H}^3$.

Under this correspondence, solutions of the SU_n-Bogomolny equations correspond to real, positive-definite bundles.

To prove this, we begin by showing that there is an analogous correspondence between non-degenerate CR-bundles on P and self-dual $SL_n(\mathbb{C})$-connections on U. The theorem arises when this correspondence operates on $\mathbb{R}_+$-invariant data.

We are aware that this type of result will be quite familiar to experts. For readers who intend to skip this section, we draw attention to the definition in Example 3, §3.1 of the line-bundles L and $\widetilde{L}$; also to the recipe given by equation (3.1) for the inverse Ward transform.

3.1 CR-bundles and self-dual connections

We begin with a description of the CR-structure on P; the reader is referred to [7] for more details of CR-structures and the $\overline{\partial}_b$-complex. On P, set

$$\Lambda^{1,0}_P = \Lambda^{1,0}_{\mathbb{CP}_3}|P, \Lambda^{0,1}_P = \Lambda^1_P \otimes \mathbb{C}/\Lambda^{1,0}_P.$$

Since P is locally defined by 1 real equation ($\mathrm{Im}\langle w, z\rangle = 0$) restriction to P cannot annihilate any non-zero $(1,0)$-form on $\mathbb{CP}_3$. It follows that $\Lambda^{1,0}_P$ and $\Lambda^{0,1}_P$ are complex vector bundles of rank 3 and 2 respectively. Let $\Lambda^{0,2}_P$ be the exterior square of $\Lambda^{0,1}_P$ and denote by $\Omega^{p,q}_P$ the space of smooth sections of $\Lambda^{p,q}_P$. Exterior differentiation of forms on P descends to give the $\overline{\partial}_b$-complex on P:

$$\Omega^{0,0}_P \xrightarrow{\overline{\partial}_P} \Omega^{0,1}_P \xrightarrow{\overline{\partial}_P} \Omega^{0,2}_P.$$

This complex is in many ways analogous to the $\overline{\partial}$-complex on $\mathbb{CP}_3$. The analogue of a holomorphic function is called a CR-function: this is a function on P annihilated by $\overline{\partial}_P$. Similarly, if $W \to P$ is a smooth vector bundle, a $\overline{\partial}_b$-operator $\overline{\partial}_W$ on W is a linear differential operator

$$\Omega^{0,0}_P(W) \xrightarrow{\overline{\partial}_W} \Omega^{0,1}_P(W);$$

that satisfies the Leibnitz rule

$$\overline{\partial}_W(fw) = (\overline{\partial}_P f)w + f\overline{\partial}_W w$$

for all functions f and sections w. Given such an operator one uses the Leibnitz rule to define the sequence of differential operators

$$\Omega^{0,0}_P(W) \xrightarrow{\overline{\partial}_W} \Omega^{0,1}_P(W) \xrightarrow{\overline{\partial}_W} \Omega^{0,2}_P(W);$$

then $\overline{\partial}_W$ is said to be *integrable* and $(W, \overline{\partial}_W)$ is said to be a CR vector bundle, if $\overline{\partial}_W^2 = 0$. If $(W', \overline{\partial}_{W'})$ is another CR vector bundle, then W is CR-equivalent to W' if there is a bundle isomorphism $\phi : W \to W'$ that intertwines the $\overline{\partial}_b$-operators:

$$\overline{\partial}_{W'} = \phi \overline{\partial}_W \phi^{-1}.$$

Example 1 If $\widetilde{W}$ is a holomorphic vector bundle over an open neighbourhood of P, then $W = \widetilde{W}|P$ is, in a natural way, a CR vector bundle on P. Indeed it follows from the definition that the restriction of a $\overline{\partial}$-operator yields a well-defined $\overline{\partial}_b$-operator over P and that integrability is preserved by restriction.

Example 2 If $\widetilde{W}$ is a holomorphic vector bundle over Z then the pull-back to P of $\widetilde{W}$ is also naturally a CR vector bundle. Conversely, if W is an $\mathbb{R}_+$-invariant CR vector bundle on P, then W descends to define a holomorphic vector bundle on Z. The main point is that pull-back yields an isomorphism $\Lambda^{0,1}_Z \to \Lambda^{0,1}_P$ at each point so the pull-back of any locally defined holomorphic function on Z will be a locally defined CR function on P.

Example 3 Let $L \to \mathbb{H}^3$ be the bundle of homogenous functions of degree 1 on U, i.e. functions f such that $f(\lambda X) = \lambda f(X)$. Denote by $\tilde{L} \to Z$ the analogous bundle of functions: $f(\lambda^{1/2} w, \lambda^{-1/2} z) = \lambda f(w, z)$. Then the bundle $\widetilde{L} \to Z$ acquires the holomorphic structure of the restriction to Z of $\mathcal{O}(1, -1) \to \mathbb{CP}_1 \times \mathbb{CP}_1$. The natural (i.e. $SO_0(1,3)$-homogeneous) holomorphic line bundles on Z have the form $\widetilde{L}^p(m)$ where p is any complex number and m any integer. On P, sections of this bundle are represented by functions of (w, z) which transform as

$$f(\mu \lambda^{1/2} w, \mu \lambda^{-1/2} z) = \mu^m \lambda^p f(w, z).$$

Notice the natural identifications $\mathcal{O}(1, 0) = \widetilde{L}^{1/2}(1)$, $\mathcal{O}(0, 1) = \widetilde{L}^{-1/2}(1)$ and, for all $2k \in \mathbb{Z}$, $\mathcal{O}_Z(k, k) = \mathcal{O}_P(2k)$.

On C, define: $A^{1,0} = \mu^* \Lambda^{1,0}_P$, $A^{0,1} = \Lambda^1_C \otimes \mathbb{C}/A^{1,0}$, $A^{0,p} = \Lambda^p A^{0,1}$, and $\mathcal{A}^{0,p} = \Gamma(C, A^{0,p})$. Exterior differentiation induces a complex

$$\mathcal{A}^{0,0} \xrightarrow{\overline{d}} \mathcal{A}^{0,1} \xrightarrow{\overline{d}} \mathcal{A}^{0,2} \xrightarrow{\overline{d}} \mathcal{A}^{0,3}$$

on C. If $V \to C$ is a smooth vector bundle, then an integrable $\overline{d}$-operator $\overline{d}_V$ on V is a linear differential operator

$$\mathcal{A}^{0,0}(V) \xrightarrow{\overline{d}_V} \mathcal{A}^{0,1}(V)$$

which satisfies the Leibnitz rule and $\overline{d}_V^2 = 0$. In this case, we shall often refer to $(V, \overline{d}_V)$ as an integrable bundle. Two integrable bundles are said to be equivalent if there is a bundle isomorphism that intertwines the $\overline{d}$-operators. The point of this definition is contained in the following

Proposition 3.2 *There is a natural 1:1 correspondence between*

(i) CR-bundles $W \to P$, modulo CR-equivalance, and

(ii) integrable bundles $(V, \overline{d}_V)$ over C, modulo equivalence.

In one direction, this correspondence is given by pulling back by μ:

$$V = \mu^* W,\ \overline{d}_V = \mu^* \overline{\partial}_W.$$

(The $\mathcal{A}$-complex was designed with precisely this in mind.) One can also go in the opposite direction, quite explicitly, by choosing a smooth section s of μ ($\mu \circ s =$ identity): putting

$$W_s = s^* V,\ \overline{\partial}_s = s^* \overline{d}_V$$

yields such an inverse. If s_1 and s_2 are 2 such sections and we put

$$W_i = s_i^* V,\ \overline{\partial}_i = s_i^* \overline{d}_V$$

then the bundle map $\phi : W_1 \to W_2$ that intertwines $\overline{\partial}_1$ and $\overline{\partial}_2$ is given by parallel transport with $\overline{d}_V$ from $s_1(P)$ to $s_2(P)$, along the fibres of μ.

Remark/Caution In the previous paragraph we have abused notation by writing μ^* for the map $\Omega_P^{0,1} \to \mathcal{A}^{0,1}$ induced by pull-back of forms by μ. (That this map is well defined follows from the definitions.) The reader is warned that we shall commit other similar notational abuses later on provided that no confusion is to be feared.

To explain now the relation between integrable $\overline{d}$-operators and connections on U, recall first that on M, the bundle of complex-valued 2-forms splits as $\Lambda^2_M \otimes \mathbb{C} = \Lambda^+_M \oplus \Lambda^-_M$, where we take Λ^+_M to be the span of

$$dX_0 \wedge dX_1 - i dX_2 \wedge dX_3, dX_0 \wedge dX_2 - i dX_3 \wedge dX_1, dX_0 \wedge dX_3 - i dX_1 \wedge dX_2.$$

If $E \to U$ is a vector bundle with a connection A, then A is called self-dual if its curvature F_A lies in $\Omega^+_M(\mathrm{End}(E))$. Finally, let us call an integrable bundle $(V, \overline{d}_V)$ *non-degenerate* if its restriction to each fibre of ν is (holomorphically) isomorphic to the trivial bundle on $\mathbb{CP}_1$.

Proposition 3.3 *There is a 1:1 correspondence between gauge-equivalence classes of self-dual connections on E and equivalence classes of non-degenerate integrable $\overline{d}$-operators on $\nu^* E$.*

Given the self-dual connection A, we can define the covariant exterior differential $d_A : \Omega^p_U(E) \to \Omega^{p+1}_U(E)$ and pull this back to C:

$$\Omega^0_C(\nu^* E) \xrightarrow{\nu^* d_A} \Omega^1_C(\nu^* E) \xrightarrow{\nu^* d_A} \Omega^2_C(\nu^* E) \xrightarrow{\nu^* d_A} \dots.$$

Since $d_A d_A = F_A$, $\nu^* d_A$ will induce an integrable $\overline{d}$-operator $\overline{d}_A$, say, on $\nu^* E$, provided that $\nu^* F_A$ lies in $\mathcal{A}^{1,0} \wedge \Omega^1_C$.

Since C is defined by $w = Xz$, $A^{1,0}$ is generated by the pull-back of dz and dXz. It is quite easy to satisfy oneself from here that the pull-back of a 2-form from U lies in $\mathcal{A}^{1,0} \wedge \Omega^1_C$ if and only if it is self-dual. Since any $\overline{\partial}$-operator that is pulled back from U satisfies the non-degeneracy condition, we have proved one half of the theorem. We omit discussion of the other half, which follows standard lines [13, 19].

We can summarize the conclusions of this subsection in the following way. Let $f : P \to U$ be a *section* of the correspondence. By this we mean that for each point γ of P, $f(\gamma)$ lies on γ. Notice that any section f determines a section $s : P \to C$ of the map μ given by $s(\gamma) = (\gamma, f(\gamma)) \in C \subset P \times U$ and that conversely if s is any section of μ then $f = \nu \circ s$ is a section of the correspondence. Then we have, according to Propositions 3.2 and 3.3 the following explicit recipe for the inverse Ward transform $\widetilde{E}$ of (E, ∇_A):

$$\widetilde{E} = (f^* E, (f^* \nabla_A)^{0,1}). \tag{3.1}$$

The freedom in the choice of f will be used in §4 to produce interesting descriptions of $\widetilde{E}$.

3.2 Homogeneous connections and hyperbolic monopoles

We now specialize to the case of $\mathbb{R}_+$-invariant data. We have already indicated (§3.1, Example 2) that there is a natural correspondence between $\mathbb{R}_+$-invariant CR-bundles on P and holomorphic vector bundles on Z. Moreover, a non-degenerate CR-bundle on P will give rise to a non-degenerate bundle on Z in this way and the Ward transform (i.e. Propositions 3.2 and 3.3) commutes with the $\mathbb{R}_+$-action so as to give a 1:1 correspondence, to be called the Hitchin–Ward transform, between $\mathbb{R}_+$-invariant self-dual connections on U and non-degenerate holomorphic bundles on Z. On the other hand it is well known that the Bogomolny equations always arise when symmetry in one direction is imposed upon the self-duality equations. In our case, a solution $(\nabla_a, i\phi)$ of the Bogomolny equations on a bundle $E \to \mathbb{H}^3$ gives rise to a self-dual connection A on $\pi^* E$ by setting

$$\nabla_A = \pi^* \nabla_a + \pi^* \phi \, d \log |X|. \tag{3.2}$$

We shall frequently use this formula, sometimes without explicitly mentioning it, to pass between $\mathbb{R}_+$-invariant self-dual connections A on U and solutions (a, ϕ) of (1.1).

3.3 Reality conditions

Because $\Lambda^{\pm}_M$ are not real subbundles, there are no real solutions of the self-duality equations on U in the ordinary sense. It does, however, make sense to require $a^* = -a$ and $\phi^* = \phi$ in (3.2) and we shall call connections of this kind *real*. There is a gauge-invariant way of expressing this which uses the inversion $I : X \mapsto X/|X|^2$ that we met in §2. Indeed since I preserves the $\mathbb{R}_+$-orbits in U, we have

$$I^*(a + \phi\, d\log|X|) = a - \phi\, d\log|X|$$

so the induced connection on $I^*E^\dagger$ (we use $E^\dagger$ to denote the conjugate dual bundle of E) is given by

$$d - a^* + \phi^*\, d\log|X|.$$

This agrees with the original connection on E iff it is real in the above sense. In invariant language, (E, ∇_A) is *real* iff there exists a bundle isomorphism

$$C : I^*E^\dagger \longrightarrow E \tag{3.3}$$

that intertwines ∇_A and the induced connection on $I^*E^\dagger$.

To find the corresponding condition on the CR-bundle $\widetilde{E}$, choose a section f of the correspondence and let $\widehat{f} = I \circ f \circ \sigma$. Since the involution σ of P induces the inversion I, $\widehat{f}$ will be another section of the correspondence, $\mathbb{R}_+$-equivariant iff and only if f is so. Let

$$\widetilde{E} = (f^*E, f^*\nabla_A), \quad \widetilde{E}' = (\widehat{f}^*E, \widehat{f}^*\nabla_A);$$

from (3.1) these are equivalent CR-bundles and each represents the inverse Ward transform of (E, ∇_A). Denote by $Q : \widetilde{E} \to \widetilde{E}'$ this equivalence. Then we have the following chain of identifications, each of which intertwines the corresponding $\overline{\partial}$-operators:

$$\sigma^*\widetilde{E}^\dagger = \sigma^* f^* I^*(I^*E)^\dagger = \widehat{f}^* I^* E^\dagger \xrightarrow{C} \widehat{f}^*E = \widetilde{E}' \xleftarrow{Q} \widetilde{E}. \tag{3.4}$$

Thus the connection A is real in the sense of (3.3) iff the CR-bundle $\widetilde{E}$ is CR-equivalent to $\sigma^*\widetilde{E}^\dagger$ and since we have been working $\mathbb{R}_+$-equivariantly, this descends to give the map $\tilde{\sigma}$ of Theorem 3.1. It follows from the definitions that if we make the same construction as in (3.4) with another section g of the correspondence, then this new CR-equivalence is intertwined with the old one (defined by the section f) under the natural CR-equivalence $g^*E \cong f^*E$. In this sense, the real structure is independent of the choice of section used to define it.

Remark The reality conditions can also be defined as in the literature (cf. [9, 11]), using the identification of the fibre $\widetilde{E}_\gamma$ as the space of covariantly constant sections of E along γ.

3.4 A comparison with 3-dimensional twistor theory

We have remarked already (§2 Remark 2) on Hitchin's approach to three dimensional twistor theory. We shall spend some time now on a comparison of this approach with ours. The results of this subsection are not used in the sequel.

The correspondence between Z and $\mathbb{H}^3$ is given by a correspondence space $C_3 \subset Z \times \mathbb{H}^3$ consisting of pairs (l, x) such that x is a point of the oriented geodesic l. Denote by μ_3 and ν_3 the restrictions to C_3 of the first and second projections. If (l, x) is a point of C_3 then l is completely determined by its velocity vector at x and it follows that C_3 is naturally diffeomorphic to $S\mathbb{H}^3$, the unit sphere bundle of $T\mathbb{H}^3$. Thus C_3 carries a natural 1-form θ given by

$$\theta_{(\xi,x)}(\dot{\xi}, \dot{x}) = \langle \xi, \dot{x} \rangle$$

where we are using $\langle \cdot, \cdot \rangle$ for the inner product on $T\mathbb{H}^3$.

We proceed now as in §3.1 to define on C_3 complex vector bundles

$$A_3^{1,0} = \mu_3^* \Lambda_Z^{1,0},\ A_3^{0,1} = \Lambda_{C_3}^1 \otimes \mathbb{C}/A_3^{1,0},\ A_3^{0,p} = \Lambda^p A_3^{0,1}$$

and $\mathcal{A}^{0,p} = \Gamma(C_3, A_3^{0,p})$. Exterior differentiation descends to define an operator $\overline{d}_3 : \mathcal{A}_3^{0,p} \to \mathcal{A}_3^{0,p+1}$ which satisfies $\overline{d}_3^2 = 0$ and the notion of an integrable bundle over C_3 is defined just as before.

Because each fibre of μ_3 is a copy of $\mathbb{R}$, one has the analogue of Proposition 3.2: there is a 1:1 correspondence between holomorphic vector bundles on Z and integrable bundles on C_3. This correspondence is induced by μ_3^* and inverted by s^* where s is any smooth section of μ_3. The Hitchin–Ward correspondence is completed by the following

Proposition 3.4 *Let $(a, i\phi)$ be a hyperbolic monopole on $E \to \mathbb{H}^3$. Then $\nu_3^* E$ is an integrable bundle when equipped with the $\overline{d}$-operator*

$$\overline{d}_{(a,\phi)} = \nu_3^* \nabla_a + \nu_3^* \phi\, \theta.$$

Conversely, every non-degenerate integrable bundle on C_3 is equivalent to one of this form.

Remark This is implicit (in the Euclidean case) in [9].

If now $s : Z \to C_3$ is a section and $f = \nu_3 \circ s$, we obtain a formula analogous to (3.1):

$$\widetilde{E} = (f^* E, [f^* \nabla_a + f^* \phi\, s^* \theta]^{0,1}). \tag{3.5}$$

We shall indicate how (3.5) can be derived from (3.1).

First identify $\mathbb{H}^3$ with the unit hyperboloid $|X|^2 = 1$ in U. Having identified $\mathbb{H}^3 \subset U$ in this way, we can lift f to an $\mathbb{R}_+$-equivariant section $F : P \to U$ given by the formula:

$$F(\gamma) = \gamma \cap \mathbb{R}_+ f(\widetilde{\pi}(\gamma)),$$

where $\widetilde{\pi} : P \to Z$ denotes the quotient map. The definition of F makes sense as γ and the $\mathbb{R}_+$-orbit of $f\widetilde{\pi}(\gamma)$ are both straight lines in a 2-plane (the $\mathbb{R}_+$-orbit of γ) and are not parallel (for one is null while the other is time-like).

Now given a hyperbolic monopole $(a, i\phi)$ on the trivial bundle $E \to \mathbb{H}^3$ we form ∇_A as in (3.2) and use F to construct the inverse Ward transform:

$$\widetilde{E} = (F^*\pi^*E, F^*(\pi^*\nabla_a + \pi^*\phi\, d\log|X|)) = (\widetilde{\pi}^* f^* E, \widetilde{\pi}^* f^* \nabla_a + \widetilde{\pi}^* f^* \phi\, d\log|F|),$$

having used the $\mathbb{R}_+$-equivariance of F. Here everything is explicitly pulled back from Z apart from $d\log|F|$. The formula (3.5) follows from the identity

$$d\log|F| = \widetilde{\pi}^* s^* \theta$$

whose proof we omit.

We conclude by remarking that there is a precise analogue of the Proposition and of (3.5) in the Euclidean case. There the twistor space is TS^2, the space of all oriented lines in $\mathbb{R}^3$, the correspondence space is $S^2 \times \mathbb{R}^3$ and this again carries a canonical 1-form θ. There is a simplification of (3.5) in this case, which arises because geodesics in $\mathbb{R}^3$ are straight lines. Indeed, the lift $s : TS^2 \to S^2 \times \mathbb{R}^3$ of any section $f : TS^2 \to \mathbb{R}^3$ is given simply by $s = (\rho, f)$ where $\rho\colon TS^2 \to S^2$ is the projection. Then the inverse Hitchin–Ward transform of a monopole $(a, i\phi)$ on a bundle $E \to \mathbb{R}^3$ is given by

$$\widetilde{E} = (f^*E, [f^*\nabla_a + f^*\phi\langle\overline{\partial} f, \rho\rangle]^{0,1})$$

where $\langle\ ,\ \rangle$ is the inner product on $T\mathbb{R}^3$.

4 Boundary conditions

We shall now impose boundary conditions on our monopoles that are sufficient to guarantee that $\widetilde{E}$ can be defined in terms of spectral curves. These conditions will be given in terms of the requirement that the various components of A should extend to ∂U as sections of various powers of L.

To explain this, suppose that f_0 is a smooth function on U homogeneous of degree 0 (i.e. $\mathbb{R}_+$-invariant) and that there exists a smooth section $f \in \Gamma(\bar{U}, L^{-1})$ such that $f_0(X) = |X| f(X)$ on U. Now $x = X/X_0$ is a good system of coordinates near ∂U and we have

$$f_0(x) = |X| f(X_0 x) = (|X|/X_0) f(x).$$

It follows that

$$f_0(x)\cosh r \to f(x)|\partial U \text{ as } r \to \infty,$$

where r is geodesic distance from $[1, 0, 0, 0]$. In these circumstances, we shall say that f_0 *extends as the section* f of L^{-1}. The beauty of this notion is that it gives an origin-independent framework for speaking about the asymptotics of exponentially decaying functions on $\mathbb{H}^3$. Extension of a function as a section of $L^{-\alpha}$ is defined analogously, as is extension of a differential form or of a section of a vector bundle defined near ∂U as a differential form or section with values in $L^{-\alpha}$.

4.1 Boundary conditions for hyperbolic monopoles

The first assumption we make is that ϕ has a limit on ∂U with constant eigenvalues $p_1 \geq p_2 \geq \ldots \geq p_n$, $\sum p_j = 0$. By (1.2), the p_j cannot all vanish, and we shall assume for simplicity 'maximal symmetry-breaking at ∞' i.e. that the p_j are all distinct. With this assumption in place, we can find an asymptotic region

$$W = \{0 < |X| \leq \varepsilon\, \mathrm{tr} X\}$$

such that over

$$\bar{W} = \{0 \leq |X| \leq \varepsilon\, \mathrm{tr} X\},$$

the eigenvalues λ_j of ϕ are distinct, $\lambda_1 > \ldots > \lambda_n$, $\lambda_j = p_j$ on ∂U. Then $E_j = \ker(\phi - \lambda_j)$ is a complex line bundle over $\bar{W}$ and since this region is homotopic to S^2, E_j has a degree $k_j \in \mathbb{Z}$. By orthogonal projection onto E_j, ∇_A defines a U_1-connection on E_j and hence an $H = (U_1)^{n-1}$-connection ∇^0 on $\oplus E_j$ preserving each summand. Identifying $E|W$ with $\oplus E_j$, we may write

$$\nabla_A = \nabla^0 + \mathrm{diag}(\lambda_j)\, d\log|X| + b^0$$

where $b^0 \in \Lambda^1(W, \mathrm{End}(E))$ has components

$$b^0_{ij} \in \Lambda^1(W, \mathrm{Hom}(E_j, E_i)) \text{ and } b^0_{ji} = (-b^0_{ij})^*.$$

The torus $H \subset SU_n$ is now the relevant symmetry group: notice that $h = \mathrm{diag}(h_i) \in H$ acts according to

$$h \cdot \nabla_A = \nabla^0 + \mathrm{diag}(\lambda_j\, d\log|X| - d\log h_j) + h \cdot b^0$$

where

$$(h \cdot b)_{ij} = (h_i h_j^{-1}) b^0_{ij}.$$

Qualitatively, the boundary conditions we shall impose are that ∇^0 extends smoothly to the boundary while the off-diagonal part b^0 tends to 0 there. To be precise we shall assume:

BC0: ∇^0 extends smoothly to $\bar{W}$;

BC1: $p_j - \lambda_j$ extends as the section ϕ_j of L^{-2} for each $j = 1, \ldots, n$, so $\lambda_j = p_j - |X|^2\phi_j$ on $\bar{W}$;

BC2: b^0_{ij} extends as the section b_{ij} of $L^{-|p_i - p_j|}$ for all i, j, so $b^0_{ij} = |X|^{|p_i - p_j|} b_{ij}$ on $\bar{W}$.

How are these conditions to be justified? Note that they are at least gauge-invariant and independent of any choice of origin in $\mathbb{H}^3$. The most satisfactory justification would be a derivation of them from the combination of the Bogomolny equations and the condition of *finite energy*: $\int_{\mathbb{H}^3} |F_a|^2 + |\nabla_a \phi|^2 < \infty$ (cf. [17, pp. 108–110] and [20, Appendix D] for results of this kind for Euclidean monopoles). We give a partial justification which explains the powers of L that appear, in the next subsection. Another partial justification is that these conditions are the natural ones required for the next result, which is necessary for the definition of spectral data in §5. Furthermore, the conditions are satisfied by any integral hyperbolic monopole (S^1-invariant instanton (cf. Introduction)); it follows that non-trivial solutions of the Bogomolny equations that satisfy BC0–BC2 do indeed exist (cf. also Remark (2) below).

We begin by reformulating BC2 as follows. Let b_+ be the lower triangular part of b: $(b_+)_{ij} = b_{ij}$ if $i > j$, $(b_+)_{ij} = 0$ otherwise; and define the upper-triangular part b_- similarly. Then it follows from the definitions that

$$\nabla_A = \nabla^0 + \operatorname{diag}(p_j - |X|^2\phi_j)\, d\log|X| + \xi^{-1} \cdot b_+ + \xi \cdot b_- \tag{4.1}$$

where

$$\xi = \operatorname{diag}(|X|^{p_1}, \ldots, |X|^{p_n})$$

lies in the complexification of H. Before giving the main result of this section we must introduce the real functions

$$\theta = \langle z, w\rangle / \langle z, z\rangle, \quad \theta' = \langle w, z\rangle / \langle w, w\rangle$$

on P. Then $\overline{\partial} + \overline{\partial}\log\theta$ and $\overline{\partial} - \overline{\partial}\log\theta'$ both define the holomorphic structure of $\widetilde{L}$ on $\mathbb{C} \times Z$. We also define 2 maps $f_0, f_\infty : P \to \partial U$ by the following formulae:

$$f_0(w, z) = ww^*/\langle w, w\rangle, \; f_\infty(w, z) = \sigma z\, \sigma z^*/\langle z, z\rangle.$$

These descend to define maps $Z \to S^2_\infty$ that will be denoted by the same symbols. Thus f_0 and f_∞ may be thought of as 'ideal sections' of the correspondence that assign to an oriented geodesic l in $\mathbb{H}^3$ its end-points. The next theorem shows that they can almost be used in the formula (3.1) provided that the monopole satisfies BC0–BC2.

Theorem 4.1 *Let $(a, i\phi)$ be a solution of the SU_n-Bogomolny equations which satisfies BC0, BC1 and BC2. Then the corresponding holomorphic bundle is represented by each of*

$$\widetilde{E} = (f_0^* E, \overline{\partial}_0) \text{ and } \widetilde{E}' = (f_\infty^* E, \overline{\partial}_\infty),$$

where

$$\overline{\partial}_0 = f_0^*(\nabla^0 + b_+) - diag(p_j)\,\overline{\partial} \log \theta' \text{ and } \overline{\partial}_\infty = f_\infty^*(\nabla^0 + b_-) + diag(p_j)\overline{\partial} \log \theta.$$

Hence there are 2 holomorphic filtrations

$$0 = \widetilde{E}_0 \subset \widetilde{E}_1 \subset \cdots \subset \widetilde{E}_n = \widetilde{E} \text{ and } 0 = \widetilde{E}'_0 \subset \widetilde{E}'_1 \subset \ldots \subset \widetilde{E}'_n = \widetilde{E}'$$

given by

$$\widetilde{E}_j = f_0^*(E_{n+1-j} \oplus \ldots \oplus E_n) \text{ and } \widetilde{E}'_j = f_\infty^*(E_1 \oplus \ldots E_j) \tag{4.2}$$

and natural isomorphisms

$$\widetilde{E}_{n-j+1}/\widetilde{E}_{n-j} = \widetilde{L}^{p_j+k_j/2}(k_j) \text{ and } \widetilde{E}'_j/\widetilde{E}'_{j-1} = \widetilde{L}^{p_j+k_j/2}(-k_j) \text{ for } j = 1, \ldots, n.$$

Proof For $0 < t < \infty$ define an $\mathbb{R}_+$-equivariant section $f_t : P \to U$ by setting

$$f_t = f_0/\theta' + (t\theta) f_\infty.$$

Let

$$\widetilde{E}_t = (f_t^* E, \overline{\partial}_t), \;\; \overline{\partial}_t = [f_t^* \nabla_A]^{0,1}. \tag{4.3}$$

We know from (3.1) that $\widetilde{E}_t$ represents the holomorphic bundle over Z which corresponds to ∇_A and from the proof of Proposition 3.2 that the equivalence $\widetilde{E}_t \to \widetilde{E}_s$ is given by parallel transport along the fibres of μ with $\nu^* \nabla_A$.

The first idea, to let $t \to 0$ or $t \to \infty$ in (4.3), with ∇_A given by (4.1), does not work because of the term in $d \log |X|$ which is ill-behaved as $t \to 0, \infty$. However, we can find a diagonal gauge transformation which eliminates this bad term and then the limits can indeed be taken.

Let us begin by using the f_t to identify $P \times \mathbb{R}_+$ with C so that $\mu : C \to P$ goes over to the first projection $\mathrm{pr}_1 : P \times \mathbb{R}_+ \to P$. This is done by mapping (γ, t) to $(\gamma, f_t(\gamma)) \in P \times U$. Then the map $f : P \times \mathbb{R}_+ \to U$ given by $f(\gamma, t) = f_t(\gamma)$ corresponds to ν. Introduce also $j_t : P \to P \times \mathbb{R}_+$, $j_t(\gamma) = (\gamma, t)$ so that $f_t = f \circ j_t$.

Notice that

$$|f_t| = \sqrt{t}\,\theta, \mathrm{tr} f_t = \frac{\theta}{\delta^2}(1 + t\delta^2) \tag{4.4}$$

where $\delta^2 = \theta\theta'$. We remark that δ^2 extends smoothly from Z to $\mathbb{CP}_1 \times \mathbb{CP}_1$ and $0 \le \delta \le 1$ with $\delta(w,z) = 0$ if and only if (w,z) lies on the anti-diagonal $\bar{\Delta}$. From (4.4),

$$\frac{|f_t|}{\mathrm{tr} f_t} = \delta \left(\sqrt{t}\,\delta + \frac{1}{\sqrt{t}\,\delta} \right)^{-1}$$

so that

$$\frac{|f_t|}{\mathrm{tr} f_t} \le \frac{\delta}{2} \text{ for all } t \text{ and } \frac{|f_t|}{\mathrm{tr} f_t} \to 0 \text{ as } t \to 0, \infty. \tag{4.5}$$

It follows that there exist $t_0, t_1 > 0$ such that $f^{-1}(W)$ contains $P \times (0, t_0]$ and $P \times [t_1, \infty)$.

From (4.1)

$$\begin{aligned} f^*\nabla_A = & \\ & f^*\nabla^0 + \mathrm{diag}(p_j - t\theta^2 f^*\phi_j) \left(\frac{dt}{2t} + d\log\theta \right) \\ & + \mathrm{diag}(t^{p_j/2}\theta^{p_j}) \cdot f^*b_- + \mathrm{diag}(t^{-p_j/2}\theta^{-p_j}) \cdot f^*b_+. \end{aligned} \tag{4.6}$$

In order to analyze the limiting behaviour of this connection it is convenient to trivialize f^*E in the t direction. Then we shall be able to think of the $\overline{\partial}_t$ as a family of operators on a fixed bundle over P, and parallel transport as defining a family of automorphisms of this same fixed bundle. It is natural to carry out such a trivialization by using parallel transport with $f^*\nabla^0$ from the boundary along the fibres of pr_1. This preserves the eigenbundles of ϕ and the boundary values of all pulled-back quantities on the RHS of (4.6). We shall abuse notation by thinking of (4.6) as now giving $f^*\nabla_A$ in this new gauge.

Having done this, the parallel-transport operator $Q_0(s,t)$ is defined by

$$\left(\frac{\partial}{\partial t} + C_0 \right) Q_0(s,t) = 0, \; Q_0(s,s) = 1 \tag{4.7}$$

where

$$C_0 = \mathrm{diag}(p_j/2t - \theta^2 f^*\phi_j/2) + \imath_t[\mathrm{diag}(t^{p_j/2}\theta^{p_j}) \cdot f^*b_- + \mathrm{diag}(t^{-p_j/2}\theta^{-p_j}) \cdot f^*b_+].$$

(Here $\imath_t$ denotes interior multiplication with $\partial/\partial t$.) As indicated above, $Q_0(s,t)$ is to be thought of as an automorphism of f_0^*E if $0 < s, t < t_0$ and of f_∞^*E if $t_1 < s, t$. From standard properties of first-order ordinary differential equations (cf. Appendix), $Q_0(s,t)$ will have a good limit as $t \to 0^+$ provided that $|C_0|$ is integrable on $(0, s)$. Similarly, integrability of $|C_0|$ on (s, ∞) will ensure good behaviour as $t \to \infty$. Thus the $p_j/2t$-term will cause $Q_0(s,t)$ to diverge as $t \to 0$ or ∞. (The p_j cannot all be zero unless $|\phi|$ is identically zero (cf. the Introduction.) On the other hand, this bad term is easily removed by changing gauge by $h = \mathrm{diag}(t^{p_j/2})$:

$$h \cdot f^*\nabla_A =$$
$$f^*\nabla^0 + \mathrm{diag}(p_j) d\log\theta - \mathrm{diag}(\theta^2 f^*\phi_j)\left(\frac{dt}{2} + t\, d\log\theta\right)$$
$$+\mathrm{diag}(t^{p_j}\theta^{p_j}) \cdot f^*b_- + \mathrm{diag}(\theta^{-p_j}) \cdot f^*b_+. \qquad (4.8)$$

Let us denote by Q the associated parallel transport operator: it is defined by the equation $[\partial/\partial t + C]Q = 0$ where

$$C = -\mathrm{diag}(\theta^2 f^*\phi_j/2) + \imath_t[\mathrm{diag}(t^{p_j}\theta^{p_j}) \cdot f^*b_- + \mathrm{diag}(\theta^{-p_j}) \cdot f^*b_+].$$

The terms in C have the form $\theta^2 f^*\phi$, $(t\theta)^\alpha f^*b_-$, or $\theta^\alpha f^*b_+$ where $\phi \in \Omega^0(\bar{W}, L^{-2})$, $b_\pm \in \Omega^1(\bar{W}, L^{-\alpha})$ and $\alpha = |p_i - p_j|$ is strictly positive. [Here we are writing $b_\pm$ for $(b_\pm)_{ij}$ to simplify notation.] We shall explain how to estimate the integrals of the b-terms, the estimate of the ϕ-term being simpler.

For a fixed geodesic γ, the integrals we have to estimate can be interpreted inside U as

$$\theta^\alpha \int_{W\cap\gamma} t^\alpha b_- \text{ and } \theta^\alpha \int_{W\cap\gamma} b_+.$$

Let Π be the time-like 2-plane that is the $\mathbb{R}_+$-orbit of γ and introduce coordinates u, v on Π via

$$(u, v) \mapsto u f_0 + v f_\infty.$$

In these coordinates, because it is horizontal, $b_\pm$ takes the form $\beta_\pm(u, v)(u\, dv - v\, du)$ where β is smooth near $u = 0$ and $v = 0$ and homogeneous of degree $-\alpha - 2$. On the other hand, γ is given by $u = 1/\theta', v = t\theta$, so the integrals become

$$\int \beta_-(1/\theta', t\theta)\frac{(t\theta)^\alpha\theta}{\theta'}\, dt$$

and

$$\int \beta_+(1/\theta', t\theta)\frac{\theta^{1+\alpha}}{\theta'}\, dt.$$

Now use the homogeneity of $\beta_\pm$ to take out a factor of $(1 + t\delta^2)/\theta'$ and our integrals become

$$\int \widehat{\beta_-}(t)\frac{\delta^{2+2\alpha}t^\alpha}{(1 + t\delta^2)^{2+\alpha}}\, dt \text{ and } \int \widehat{\beta_+}(t)\frac{\delta^{2+2\alpha}}{(1 + t\delta^2)^{2+\alpha}}\, dt \qquad (4.9)$$

where

$$\hat{\beta}(t) = \beta\left(\frac{1}{1 + t\delta^2}, \frac{t\delta^2}{1 + t\delta^2}\right) \qquad (4.10)$$

is bounded for $t < t_0$ and $t > t_1$. It follows that (4.9) is bounded on $(0, t_0)$ and (t_1, ∞). Since the ϕ-terms are dealt with similarly it follows that

$$\lim_{t\to 0} Q(t_0, t) \text{ and } \lim_{t\to\infty} Q(t_1, t)$$

both exist. The limits $Q(t_0, 0)$ and $Q(t_1, \infty)$ will moreover define smooth automorphisms of f_0^*E, f_∞^*E because C is smooth up to the boundary. We have now proved that the limiting $\overline{\partial}$-operators, obtained by pulling back $h \cdot f^*\nabla_A$ by j_t and letting t go to zero or ∞ define the correct holomorphic structure. What remains is to evaluate these limits and show that they lead to the holomorphic filtrations given in the Theorem.

To do this, note

$$f_t = f_0/\theta' + O(t) \text{ as } t \to 0, f_t = t\theta(f_\infty + O(t^{-1})) \text{ as } t \to \infty$$

and (as above) the various terms on the RHS of (4.8) that do not involve dt have one of the following three forms: $t\theta^2 f^*\phi$, $(t\theta)^\alpha f^*b$ and $\theta^\alpha f^*b$ ($\alpha > 0$). Using the asymptotic forms of f_t and the given homogeneities of the quantities involved, one checks that

$$\begin{aligned} t\theta^2 f_t^*\phi \to 0 &\quad \text{as} \quad t \to 0, \infty; \\ (t\theta)^\alpha f_t^*(b) \to 0 &\quad \text{as} \quad t \to 0, \\ (t\theta)^\alpha f_t^*(b) \to f_\infty^* b &\quad \text{as} \quad t \to \infty; \\ \theta^\alpha f_t^*(b) \to \delta^{2\alpha} f_0^* b &\quad \text{as} \quad t \to 0, \\ \theta^\alpha f_t^*(b) \to 0 &\quad \text{as} \quad t \to \infty. \end{aligned} \tag{4.11}$$

Putting the pieces together we find

$$h \cdot f_t^*\nabla_A \to f_0^*\nabla^0 + \operatorname{diag}(p_j) d\log\theta + \operatorname{diag}(\delta^{-p_j}) \cdot f_0^* b_+ \text{ as } t \to 0$$

and changing gauge by $\operatorname{diag}(\delta^{p_j})$ now yields the first statement of the theorem. Similarly, taking the limit as $t \to \infty$ of $h \cdot f_t^*\nabla_A$ yields the second statement.

From the triangular structure of $\overline{\partial}_0$ and $\overline{\partial}_\infty$, it is clear that the subbundles defined in (4.2) are holomorphic. The induced holomorphic structure on the quotient $\widetilde{E}_{n-j+1}/\widetilde{E}_{n-j}$ is given by the diagonal term $f_0^*\nabla_j^0 - p_j\overline{\partial}\log\theta'$ of $\overline{\partial}_0$ and similarly that on $\widetilde{E}'_j/\widetilde{E}'_{j-1}$ is given by $f_\infty^*\nabla_j^0 + p_j\overline{\partial}\log\theta$. Now f_0 is the restriction of the first projection $\mathbb{CP}_1 \times \mathbb{CP}_1$, so by choosing the orientation of S^2_∞ so that f_0 is orientation-preserving, the holomorphic structure defined by $f_0^*\nabla_j^0$ is just $\mathcal{O}(k_j, 0) = \widetilde{L}^{k_j/2}(k_j)$. [Recall that the complex line-bundle of degree k on S^2 has a unique holomorphic structure, defined by the $(0,1)$-part of any connection on it.] Since $-p_j\overline{\partial}\log\theta'$ contributes the twist $\widetilde{L}^{p_j}$ we obtain the desired isomorphism $\widetilde{E}_{n-j+1}/\widetilde{E}_{n-j} = \widetilde{L}^{p_j+k_j/2}(k_j)$.

Similar considerations apply to the diagonal part of $\overline{\partial}_\infty$. The only point that requires care is that, with the orientations chosen as above, f_∞ is orientation-*reversing*. (It is the composition of the second projection with the antipodal map.) Thus the pull-back of E_j by f_∞ has degree $-k_j$ and holomorphic structure $\mathcal{O}(0,-k_j) = \widetilde{L}^{k_j/2}(-k_j)$. As in the previous case, the term $p_j\overline{\partial}\log\theta$ has the effect of tensoring with $\widetilde{L}^{p_j}$. □

Remarks (1) For SU_2 monopoles, we have $p_1 = p = -p_2 > 0$, $k_1 = -k_2 = k > 0$ and the 2 filtrations become exact sequences

$$0 \to \widetilde{L}^{-p-k/2}(-k) \to \widetilde{E} \to \widetilde{L}^{p+k/2}(k) \to 0$$

and

$$0 \to \widetilde{L}^{p+k/2}(-k) \to \widetilde{\widetilde{E}} \to \widetilde{L}^{-p-k/2}(k) \to 0.$$

(2) The only part of the description of $(f_0^*E, \overline{\partial}_0)$ that is *not* pulled back from the first factor of Z is the $\overline{\partial}\log\theta'$ term. If $p_j \in \mathbb{Z}$, so the monopole is *integral*, we can write $\overline{\partial}_0$ as

$$\lambda[f_0^*(\nabla^0) + \lambda^{-1}\cdot f_0^* b_+]\lambda^{-1}$$

where

$$\lambda = \mathrm{diag}(\theta'^{p_j})$$

is viewed as a bundle map to $\oplus\mathcal{O}(k_j + p_j, -p_j)$ and the operator in square brackets is thought of as a $\overline{\partial}$-operator on this bundle. One calculates that $\lambda^{-1}\cdot b_+$ extends smoothly through the anti-diagonal $\theta' = 0$ of $\mathbb{CP}_1 \times \mathbb{CP}_1$. Hence $\widetilde{E}$ extends from Z to $\mathbb{CP}_1 \times \mathbb{CP}_1$ and, if $n = 2$, we recover the splittings found by Atiyah [1, §4]. (Of course similar remarks apply to $\widetilde{E}'$.)

In this sense, it is the fact that the non-integral powers of $\widetilde{L}$ do not extend to $\mathbb{CP}_1 \times \mathbb{CP}_1$ that is the only obstruction to the extension of $\widetilde{E}$ in the non-integral case.

(3) The theorem clearly displays a relation between the twistor description of a hyperbolic monopole satisfying BC0–BC2 and the boundary data of the monopole. The Hitchin–Ward transform emerges as a non-linear integral transform that reconstructs the monopole from its boundary values. We can say precisely what boundary data are needed for the twistor description. From the formula for $\overline{\partial}_0$, we see that in addition to the p_j and k_j the relevant data are $(\nabla^0|S^2_\infty)^{0,1}$ and $(b_+|S^2_\infty)^{0,1}$. (Here the $(0,1)$-parts are calculated relative to the orientation of S^2_∞ that appeared in the proof of Theorem 4.1.) This observation is particularly intriguing in view of Theorem 5.1 of [8], which asserts that any integral hyperbolic SU_2-monopole is determined by $\nabla^0|S^2_\infty$. For such monopoles, then, our result appears to be weaker than theirs.

(4) The filtrations correspond to each other under the real structure. They can also be constructed by a method more analogous to that used by Hitchin and Murray [11] for Euclidean monopoles, in terms of decaying solutions of an ODE along oriented geodesics.

4.2 Justification of the boundary conditions

As an appendix to this section, we look here at the Bogomolny equations over W, with $(a, i\phi)$ in the form (4.1). It is simplest to calculate in the upper half-space model of $\mathbb{H}^3$. Suppose that $E = \oplus E_j$ is trivialized over some strip $0 \le t \le \varepsilon$ by parallel transport with ∇^0 along vertical straight lines, so that

$$\nabla^0 = d + \mathrm{diag}(a_j d\bar{u} - \bar{a}_j du)$$

and the full connection is given by

$$\nabla_a = \nabla^0 + b^0.$$

Then

$$F_a = F^0 + d^0 b^0, \;\; \nabla_a \phi = \mathrm{diag}(d\lambda_j) + [b^0, \mathrm{diag}(\lambda_j)].$$

Now in these coordinates, we have

$$*(dt \wedge du) = it\, du, \; *(dt \wedge d\bar{u}) = -it\, d\bar{u}, \; *(du \wedge d\bar{u}) = 2it\, dt.$$

It follows that if ∇^0 extends to the boundary and b^0 decays, the leading term in the Bogomolny equation is diagonal and gives

$$d\lambda = t[F^0_{tu} du - F^0_{t\bar{u}} d\bar{u} + 2F^0_{u\bar{u}} dt].$$

Since the RHS vanishes at $t = 0$, it follows (as we have anticipated) that the eigenvalue λ is constant on S^2_∞. The dt-component gives $\partial\lambda/\partial t = 2tF^0_{u\bar{u}}$ and integrating this gives $\lambda = p + t^2 F^0_{u\bar{u}} + \ldots$; since $t = |X|$ in this model, this explains the exponent -2 that appears in BC1. It gives more, namely the interpretation (up to constant factors) of $\mathrm{diag}(\phi_j)|\partial U$ as the curvature of $\nabla^0|\partial U$.

Now consider a typical off-diagonal term b^0_{ij} and assume that it takes the form $b^0_{ij} = t^r b_{ij} = t^r(b_u du + b_{\bar{u}} d\bar{u} + b_t dt)$ where b_{ij} is smooth at $t = 0$. Then the leading order term in $d^0 b^0$ is $rt^{r-1} dt \wedge b_{ij}$, while $[b^0, \phi]_{ij} = -(p_i - p_j) b^0_{ij}$. Hence the ij component of the Bogomolny equations is, to leading order,

$$rt^r(b_u du - b_{\bar{u}} d\bar{u}) = -(p_i - p_j) t^r (b_u du + b_{\bar{u}} d\bar{u})$$

from which $r = |p_i - p_j|$, the exponent that appears in BC2.

5 Spectral data

We have now constructed a holomorphic bundle $\widetilde{E}$ over Z with two fitrations $\widetilde{E}_r$ and $\widetilde{E}'_r$ which satisfy a reality condition that there is an anti-holomorphic map $\sigma\colon \widetilde{E} \to \widetilde{E}^*$ covering the real structure σ on Z and such that

$$\sigma(\widetilde{E}_i) = (\widetilde{E}'_{n-i})^{\perp}$$

for all $i = 1, \dots n$. Here if W is a subspace of a vector space V we denote by $W^\perp$ the subspace of V^* of linear functionals that vanish on W. The situation is entirely analogous with the Euclidean case and we can define the spectral data in the same way.

We define the r-th spectral curve S_r to be the subvariety of Z where the map

$$\psi_r \colon \Lambda^r(\widetilde{E}'_r) \to \Lambda^r(\widetilde{E}/\widetilde{E}_{n-r})$$

vanishes for $i = 1, \dots, n-1$. From the results of the previous section we see that ψ_r is a section of $\mathcal{O}(2m_r)$ where $m_i = k_1 + \dots + k_r$ is called the rth magnetic charge of the monopole. We have

Lemma 5.1 *The spectral curve S_i is compact.*

Proof We refer to the notation of Theorem 4.1 and its proof.

If $\eta > 0$, put $Z_\eta = \{(w, z) \in Z : 0 < \delta(w, z) < \eta\}$ and recall that δ vanishes precisely on $\bar{\Delta}$. We shall prove that there exists $\eta > 0$ such that the intersection of the sub-bundles $\widetilde{E}_{n-j+1}$ and $\widetilde{E}'_{j-1}$ inside $\widetilde{E}$ is zero over Z_η. For this, we must consider again the parallel transport operator Q since it is $Q(0, \infty)$ that intertwines $\overline{\partial}_0$ and $\overline{\partial}_\infty$.

From (4.5), the geodesic in U corresponding to any point of Z_η lies entirely in W as soon as $\eta < \varepsilon$. But for such γ the function $\hat{\beta}$ of (4.10) is smooth and hence bounded by some constant K_0, say, for all t and so we have a global estimate for the integral of $\widetilde{C}_+$; from (4.9),

$$\left\| \int_0^\infty \theta^\alpha f^* b_+ \right\| \le K_0 \int_0^\infty \frac{\delta^{2+2\alpha}}{(1 + t\delta^2)^{2+\alpha}}\, dt = \frac{K_0 \delta^{2\alpha}}{1+\alpha}.$$

Now $\alpha > 0$ for the terms in C_+ so this calculation shows that the global L^1-norm of C_+ decays like $\delta^{2\alpha}$ where $\alpha = \inf_{i \ne j} |p_i - p_j|$. It follows from ODE theory (cf. Appendix) that there is a constant K such that

$$|Q(0, \infty)_+| \le K \delta^{2\alpha} \text{ over } Z_\varepsilon.$$

In other words, $Q(0, \infty)$ is approximately upper-triangular for small values of δ.

It follows that there exists $\eta > 0$ such that over Z_η, the composite

$$f_0^*(\oplus_{i=j}^n E_i) \hookrightarrow f_0^* E \overset{Q(0,\infty)}{\longrightarrow} f_\infty^* E \longrightarrow f_\infty^*(\oplus_{i=j}^n E_i)$$

is invertible for all j. By definition of the filtrations, this means that the composite

$$\widetilde{E}_{n-j+1} \to \widetilde{E} \to \widetilde{E}/\widetilde{E}'_{j-1}$$

is invertible, i.e. $\widetilde{E}_{n-j+1} \cap \widetilde{E}'_{j-1} = 0$ over Z_η. □

Lemma 5.1 has the important consequence that the section ψ_i cannot vanish and hence the magnetic charges m_i are non-negative.

The spectral curves form part of the so-called spectral data. The remainder is a decomposition of the intersection of the spectral curves S_r and S_{r+1} into two pieces. We define

$$S_{r,r+1} = \{x \in S_r \mid \dim(\widetilde{E}'_r \cap \widetilde{E}_{n-r-1}) \geq 1\}$$

and

$$S_{r+1,r} = \{x \in S_r \mid \dim(\widetilde{E}'_{r+1} \cap \widetilde{E}_{n-r}) \geq 2\}.$$

It is possible to show via some linear algebra that at least as sets $S_r \cap S_{r+1} = S_{r,r+1} \cap S_{r+1,r}$. To define $S_{r,r+1}$ and $S_{r+1,r}$ as varieties it is more convenient to use flag manifolds as was done for the Euclidean case in [18]. To avoid these technical questions we follow [11] and define a monopole to be *general* if the spectral curves S_r and S_{r+1} intersect transversally and the $S_{r,r+1}$ and the $S_{r+1,r}$ are distinct sets of points. Note that the spectral curves will intersect in $2m_r m_{r+1}$ points and the $S_{r,r+1}$ and $S_{r+1,r}$ will consist of $m_r m_{r+1}$ points interchanged by the real structure.

Let us illustrate the spectral data by discussing two examples.

Example 1 In the case of $SU(2)$ we have a rank two bundle $\widetilde{E}$ with two sub line bundles $\widetilde{E}_1$ and $\widetilde{E}'_1$. There is only one spectral curve S_1 and it is the curve of points where $\widetilde{E}_1 = \widetilde{E}'_1$. In this case the spectral data is just the spectral curve S_1.

Example 2 In the case of $SU(3)$ we have two filtrations;

$$\widetilde{E}_1 \subset \widetilde{E}_2 \subset \widetilde{E}$$

and

$$\widetilde{E}'_1 \subset \widetilde{E}'_2 \subset \widetilde{E}.$$

There are two spectral curves. The first, S_1 is the curve where $\widetilde{E}'_1 \subset \widetilde{E}_2$ and the second, S_2 where $\widetilde{E}_1 \subset \widetilde{E}'_2$. The spectral data is the division of the intersection of these two curves into two sets of points: $S_{1,2}$ and $S_{2,1}$. The condition for a point to be in S_{12} is that $\widetilde{E}'_1 = \widetilde{E}_1$ and the condition for a point to be in S_{21} is that $\widetilde{E}'_2 = \widetilde{E}_2$.

We conclude this section by showing that the spectral data determines the monopole. This was first done by Hitchin for the case of $SU(2)$ in [9] and by Murray for the case of $SU(n)$ in [18]. The proof in [18] does not generalise well to hyperbolic space but there is another proof in [15] that does generalise. Because it generalises so readily to the case at hand we will not repeat it in detail. Instead we will explain how it applies to the $SU(2)$ case and the sketch the general $SU(n)$ case relying on [15] for details and proofs.

Consider the short exact sequence of sheaves.

$$0 \to \widetilde{E} \to \widetilde{E}/\widetilde{E}_1 \oplus \widetilde{E}/\widetilde{E}_1' \to \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1) \to 0$$

Notice that we have to think about this as a sequence of sheaves. The first two sheaves are actually bundles but the last is a sheave supported on S_1 because at points not on S_1 the fibres of the lines bundles satisfy $\widetilde{E} = \widetilde{E}_1' \oplus \widetilde{E}_1$. We also need to specify the maps. The first just maps an element a to a pair $(a + \widetilde{E}_1', a + \widetilde{E}_1)$. The second takes a pair $(a_1, a_2) \in \widetilde{E}/\widetilde{E}_1' \oplus \widetilde{E}/\widetilde{E}_1$ and maps it to $\beta_1(a_1) - \alpha_2(a_2) \in \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1)$ where

$$\alpha_2 \colon \widetilde{E}/\widetilde{E}_1' \to \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1)$$

and

$$\beta_1 \colon \widetilde{E}/\widetilde{E}_1 \to \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1).$$

Clearly the composition of these two maps is zero. It is completely straightforward to adapt Proposition 1.12 of [15] to prove that this sequence is exact.

The point of this construction is that the part of the sequence

$$\widetilde{E}/\widetilde{E}_1' \oplus \widetilde{E}/\widetilde{E}_1 \to \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1) \tag{5.1}$$

is determined entirely by the spectral curve and known line bundles. Hence $\widetilde{E}$, being the kernel of this map, is determined by the spectral curve. The first of the sheaves in (5.1) is a direct sum of vector bundles determined by Theorem 4.1 where it is shown that

$$\widetilde{E}/\widetilde{E}_1' = \widetilde{L}^{p_2 + k_2/2}(-k_2) \text{ and } \widetilde{E}/\widetilde{E}_1 = \widetilde{L}^{p_1 + k_1/2}(k_1).$$

For the second sheaf in (5.1) recall that the spectral curve S_1 is defined by the vanishing of the map

$$\psi_1 \colon \widetilde{E}_1' \to \widetilde{E}/\widetilde{E}_1$$

and hence the sheaf $\mathcal{O}_{S_1}(\widetilde{E}/\widetilde{E}_1)$ of sections of $\widetilde{E}/\widetilde{E}_1$ over S_1 fits naturally into a short exact sequence of sheaves:

$$0 \to \widetilde{E}_1' \stackrel{\psi_1}{\to} \widetilde{E}/\widetilde{E}_1 \to \mathcal{O}_{S_1}(\widetilde{E}/\widetilde{E}_1) \to 0.$$

But we also have the exact sequence of sheaves

$$0 \to \widetilde{E}_1' \stackrel{\psi_1}{\to} \widetilde{E}/\widetilde{E}_1 \to \widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1) \to 0$$

so we must have

$$\widetilde{E}/(\widetilde{E}_1' + \widetilde{E}_1) = \mathcal{O}_{S_1}(\widetilde{E}/\widetilde{E}_1) = \mathcal{O}_{S_1}(\widetilde{L}^{p_1 + k_1/2}(k_1)).$$

The final thing we need to determine is the map in (5.1). Under the identifications we have already given the map

$$\alpha_1 \colon \widetilde{E}/\widetilde{E}_1 \to \widetilde{E}/(\widetilde{E}'_1 + \widetilde{E}_1)$$

is just the restriction map

$$\widetilde{L}^{p_1+k_1/2}(k_1) \to \mathcal{O}_{S_1}(\widetilde{L}^{p_1+k_1/2}(k_1))$$

and the map

$$\beta_1 \colon \widetilde{E}/\widetilde{E}'_1 \to \widetilde{E}/(\widetilde{E}'_1 + \widetilde{E}_1)$$

has become a map

$$\widetilde{L}^{p_2+k_2/2}(-k_2) \to \mathcal{O}_{S_1}(\widetilde{L}^{p_1+k_1/2}(k_1)).$$

This last map is restriction to S_1 and then multiplication by an isomorphism

$$\mathcal{O}_{S_1}(\widetilde{L}^{p_2+k_2/2}(-k_2)) \to \mathcal{O}_{S_1}(\widetilde{L}^{p_1+k_1/2}(k_1)).$$

This isomorphism is induced by a holomorphic section ρ_1 of

$$(\widetilde{L}^{p_2+k_2/2}(-k_2))^* \otimes \widetilde{L}^{p_1+k_1/2}(k_1) = \widetilde{L}^{p_1-p_2+(k_1-k_2)/2}.$$

over S_1. Because this is a trivial bundle and the curve S_1 is compact the section ρ_1 is determined unqiuely up to scale and the choice of scale makes no difference to the final isomorphism class of $\widetilde{E}$. In terms of the original bundle the section ρ_1 arises because of the equality of $\widetilde{E}'_1$ and $\widetilde{E}_1$ over S_1. This equality induces an isomorphism of line bundles

$$\widetilde{L}^{p_2+k_2/2}(-k_2) = \widetilde{E}/\widetilde{E}'_1 \to \widetilde{E}/\widetilde{E}_1 = \widetilde{L}^{p_1+k_1/2}(k_1)$$

over S_1.

For the general case consider the sequence of sheaves:

$$0 \to \widetilde{E} \to \bigoplus_{r=1}^{n} \widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}_{n-r}) \to \bigoplus_{r=1}^{n-1} \widetilde{E}/(\widetilde{E}'_r + \widetilde{E}_{n-r}) \to 0.$$

The first map here is the obvious sum of projections and the second is an alternating sum of the form

$$(a_1, \ldots, a_n) \mapsto (\beta_1(a_1) - \alpha_2(a_2), \beta_2(a_2) - \alpha_3(a_3), \ldots, \beta_{n-1}(a_{n-1}) - \alpha_n(a_n))$$

where the maps α_r and β_r are the projections

$$\alpha_r \colon \widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}_{n-r}) \to \widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}_{n-r+1})$$

and

$$\beta_r\colon \widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}_{n-r}) \to \widetilde{E}/(\widetilde{E}'_r + \widetilde{E}_{n-r}).$$

Clearly 5.2 is a complex and it is shown in [15] that it is a short exact sequence. Again it follows that the bundle $\widetilde{E}$ is determined as the kernel of

$$\bigoplus_{r=1}^{n} \widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}_{n-r}) \to \bigoplus_{r=1}^{n-1} \widetilde{E}/(\widetilde{E}'_r + \widetilde{E}_{n-r}). \tag{5.2}$$

We refer the reader to [15] for the details of the proof that (5.2) is determined by the spectral data and just summarise here the results. The first term in (5.2) is a sum of the sheaves

$$\widetilde{E}/(\widetilde{E}'_{r-1} + \widetilde{E}'_{n-r}) = \widetilde{L}^{p_r + k_r/2}(m_{r-1} + m_r) \otimes \mathcal{I}(S_{r-1,r})$$

where $\mathcal{I}(S_{r-1,r})$ is the ideal sheaf of the points $S_{r-1,r}$; that is the sheaf of functsions vanishing at the points $S_{r-1,r}$. The second term is a sum of sheaves supported on the spectral curves. The general term is:

$$\widetilde{E}/(\widetilde{E}_r + \widetilde{E}'_{n-r}) = \widetilde{L}^{p_{r+1} + k_{r+1}/2}(m_r + m_{r+1})[-S_{r,r+1}]|_{S_r}$$

a sheaf supported on S_r. Here $[D]$ is the line bundle determined by the divisor D. Under these identifications the projections α_r and β_r are maps

$$\alpha_r\colon \widetilde{L}^{p_r + k_r/2}(m_r + m_{r+1}) \otimes \mathcal{I}(S_{r-1,r}) \to \widetilde{L}^{p_r + k_r/2}(m_{r-1} + m_r) \otimes [-S_{r-1,r}]|_{S_{r-1}}$$

and

$$\beta_r\colon \widetilde{L}^{p_r + k_r/2}(m_{r+1} + m_r) \otimes \mathcal{I}(S_{r-1,r}) \to \widetilde{L}^{p_{r+1} + k_{r+1}/2}(m_r + m_{r+1}) \otimes [-S_{r,r+1}]|_{S_r}.$$

The map α_r is just restriction to the spectral curve S_{r-1}. The map β_r is restriction to S_r followed by multiplication by a meromorphic section ρ_r/ψ_{r+1}. The section ρ_r is a holomorphic section of

$$\widetilde{L}^{p_{r+1} + k_{r+1}/2 - p_r - k_r/2}(m_{r-1} + m_{r+1})$$

over the spectral curve S_r. It is completely determined, up to scale, by the fact that its divisor, the set where it vanishes is $S_{r,r-1} \cup S_{r,r+1}$. So we conclude that the spectral data determines (5.2) and hence the bundle $\widetilde{E}$. In [15] they show how to construct the real structure on $\widetilde{E}$ from the spectral data so we conclude that the general monopole is determined by its spectral data.

A Some results from the theory of ODEs

In this article we have used some facts about parallel transport. These facts follow from the theory of the ODE

$$\left(\frac{\partial}{\partial t} + C(t)\right) Q(t) = 0,\ t \in I,\ Q(0) = 1$$

where $I \subset \mathbb{R}$ is an open interval containing 0 and $C : I \to \mathrm{End}(\mathbb{C}^n)$ is smooth. We shall now explain how these results are proved.

First, let $|C(t)|$ denote the point-wise L^2-norm of C and let

$$\|C\|_1 = \int_I |C|,\ \|C\|_\infty = \sup_I |C|.$$

We suppose that $\|C\|_1 < \infty$. Then by rescaling t or truncating I as appropriate, we may assume that

$$\|C\|_1 = k < 1.$$

Write $Q = 1 + h$, and define the operator T by

$$Th(t) = -\int_0^t C(s)\, ds - \int_0^t C(s)h(s)\, ds.$$

This is the starting point for the standard existence proof, for T is readily shown to be a contraction mapping on $L^\infty(I)$ and the fixed point gives the solution to our equation.

We are more concerned with *a priori* estimates for h. Since the solution h satisfies

$$h(t) = -\int_0^t C(s)\, ds - \int_0^t C(s)h(s)\, ds,$$

the obvious estimate gives

$$\|h\|_\infty \leq \|C\|_1 + \|C\|_1 \|h\|_\infty$$

and hence

$$\|h\|_\infty \leq \frac{k}{1-k}.$$

This is the key to proving that the solution exists up to the boundary of I as needed in the proof of Theorem 4.1

The other fact we needed was an estimate for h_+ given control of $\|C_+\|_1$. So suppose that

$$\|C_+\|_1 = \varepsilon.$$

Then taking the lower-triangular part of h,

$$h_+(t) = -\int_0^t C_+(s)\,ds - \int_0^t [C(s)h(s)]_+\,ds$$

and since

$$|[Ch]_+| \leq |C||h_+| + |C_+||h|$$

we obtain

$$\|h_+\|_\infty \leq \varepsilon + k\|h_+\|_\infty + \varepsilon\|h\|_\infty$$

and using our earlier estimater for $\|h\|_\infty$, this leads quickly to

$$\|h_+\|_\infty \leq \frac{\varepsilon}{(1-k)^2}.$$

This result was required in the proof of Lemma 5.1.

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