How to see in many dimensions

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What is dimension?

I will assume you have an intuitive idea of what is meant by dimension.

The surface of this screen is 2 dimensional.

The room we are in is 3 dimensional.

Dimension means the number of degrees of freedom in a problem or the number of variables required to specify the configuration of something.
Why do we need higher dimensions?

- The real world of space is 3 dimensional.
- But there is also time which makes it $3+1 = 4$ dimensional.
- Some physical theories suggest that the real world is 10 or 24 dimensional with the `extra’ dimensions squashed up very, very small.
Consider this robot arm. How many numbers do we need to specify its configuration?

\[ 1 + 1 + 1 + 1 + 1 + 1 = 6 \] dimensions

So the configuration is determined by a point in \( \mathbb{R}^6 \).

Important to understand this space to operate the arm.
Our intuition is limited by the world we evolved in.

Mathematics can help us expand our intuition and cope with situations where we have little natural intuition.

Mathematics can also give us tools for thinking about higher dimensions.
Descartes (1596–1650) introduced in two dimensions the idea that we could specify a point as two numbers \((x, y)\) relative to some axes.
This led to the idea of having more numbers than two:

- \((x, y, z)\) for points in 3-space
- \((x, y, z, t)\) for points in space-time
- \((x_1, x_2, \ldots, x_n)\) for points in an arbitrary \(n\) dimensional space we like to call \(\mathbb{R}^n\)

Notice that from the mathematical point of view there is no difference between two and three dimensions which we can visualise and 42 dimensions which we can’t visualise.
3d to 2d

A useful thing to do is to think about how we can see 3 dimensional objects in 2 dimensions and then try to use those ideas in higher dimensions.

There are two things I want to describe:

- Photograph - projection
- Contours - Morse theory
Projection

- A photograph renders a 3d scene onto a 2d scene.
- For example a cube viewed end on looks like

Mathematically this is projection from a point onto a plane.
We can do the same thing to a 4d hypercube and project it into 3d to get

Why do we know a hypercube looks like this?
Think about getting from a line to a square

Start with a line and let it move forward 1 second in time to make a hyperline or a square:

So a square has
2 + 2 = 4 vertices,
1 + 2 + 1 = 4 edges
1 face (square)
Do the same trick with a cube:

8 vertices
12 edges
6 faces
1 cube

So a hypercube consists of:

8 + 8 = 16 vertices
12 + 8 + 12 = 32 edges
6 + 12 + 6 = 24 faces
1 + 6 + 1 = 8 cubes

Remember 24 faces in a hypercube!
You can count them all here:

- $8 + 8 = 16$ vertices
- $12 + 8 + 12 = 32$ edges
- $6 + 12 + 6 = 24$ faces
- $1 + 6 + 1 = 8$ cubes
You can see them all here as we rotate the hypercube in 4d and look at the projection into 3d:
What if we don’t use pictures?

An n-cube is:

\[ C_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [0, 1], i = 1, \ldots, n \} \]

Examples:

- \( C_1 = [0, 1] \) is a line
- \( C_2 = \{ (x, y) \mid x \in [0, 1], y \in [0, 1] \} \) is a square
- \( C_3 \) is a cube

etc ...
Faces of an n-cube

We saw that a cube has 8 vertices, 12 edges and 6 faces.

Let's call these 0-faces, 1-faces and 2-faces instead of vertices, edges and faces.

So for example in a 2-cube or square we have 0-faces (vertices) given by:

\[
\{(0, 0)\} \hspace{1cm} \{(0, 1)\} \hspace{1cm} \{(1, 0)\} \hspace{1cm} \{(1, 1)\}
\]

It also has 1-faces or edges give by:

\[
\{(0, x_2) \mid x_2 \in [0, 1]\} \hspace{1cm} \{(1, x_2) \mid x_2 \in [0, 1]\} \\
\{(x_1, 0) \mid x_2 \in [0, 1]\} \hspace{1cm} \{(x_1, 1) \mid x_2 \in [0, 1]\}
\]
Counting the k-faces in an n-cube

A k-face is a subset of a n-cube where k of the components are allowed to vary but the other n-k are fixed to be 0 or 1. For example a 2-face in a 5-cube:

\{(1, x_2, 0, x_4, 1) \mid x_2, x_4 \in [0, 1]\}

This is k choices of positions for the variables and n-k choices of 0s and 1s so the number of k-faces in an n-cube is:

\[ \binom{n}{k} 2^{n-k} \]

Check by counting faces in the hypercube. So that is 2-faces in a 4-cube giving ..

\[ \binom{4}{2} 2^2 = 24 \]
Projection has a number of problems:

- Lose information that is hidden at the back.
- Not so good in higher dimensions.
- Lose track of symmetries.
Contours - Morse Theory

Another way to describe a 3d scene in 2d is by contours:
Imagine a torus sitting in 3d and we plot the height above the ground on it. In other words the level sets of the height function. This is similar to drawing contour lines.
We are interested in how the surface changes as we move from the bottom to the top.

Can we build the torus from the bottom up by sticking all the contours back together?
Morse theory is interested in when the **topology** of the surface changes.

Deforming and stretching does not change the topology. For example:

have the same topology as do
and

but not

and

and
Look at the picture again and try and spot where the topology changes:
The topology only changes at four places:

These points are where the derivative of $h$ or $\nabla h$ vanishes. Usually called the **critical points** of $h$. 
The reason for this is that $\nabla h$ defines a vector field which shows us how to deform the surface as $h$ increases.

So when $\nabla h$ is not zero the surface just stretches in the direction of $\nabla h$. No change in the topology.

When $\nabla h$ is zero, at a critical point, the topology can change.
At the critical points we do what a topologist calls adding a cell or adding a handle. For example at the second critical point the change in the surface is the same as adding the half-circle handle as below.

How does the topology change?

The size of the cell you add depends on the number of downward directions, or negative eigenvalues of the Hessian of $h$. 

$$C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i^2 \in [0, 1], i = 1, \ldots, n\}$$

$$C_1 = [0, 1]$$

$$C_2 = \{(x, y) | x^2 \in [0, 1], y^2 \in [0, 1]\}$$

$$C_3 = \{(x_1, \ldots, x_n) \} \begin{cases} (1, x_2, 0, x_4, 1) | x_2, x_4^2 \in [0, 1] \\ (0, x_2) | x_2^2 \in [0, 1] \\ (1, x_2) | x_2^2 \in [0, 1] \end{cases}$$

$$\nabla h \cdot \nabla h = 0$$

$X: X! \mathbb{R}^n(x, y, z, t)(x_1, x_2, \ldots, x_n)$

$$R^6_0^1$$
Now we have the theory we can forget about the pictures.

You have a space $X$ and a function $h: X \to \mathbb{R}$ (of the right kind) from which you can deduce many facts about the topology of $X$.

Morse theory turns out to be a powerful tool for understanding higher dimensional (and some infinite dimensional) spaces.

Morse theory also has applications to problems in computer vision.
Thanks to

- Wikipedia article on Morse Theory

- Gert Vegter for letting me use his graphics from
  - http://graphics.stanford.edu/courses/cs468-02-fall/schedule.html

- John Stembridge for letting me use his pictures of 4d polyhedra from
  - http://www.math.lsa.umich.edu/~jrs/coxplane2.html