

BUNDLE GERBES AND FIELD THEORY

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ABSTRACT. This paper reviews recent work on a new geometric object called a bundle gerbe and discusses examples arising in quantum field theory. One application is to an Atiyah-Patodi-Singer index theory construction of the bundle of fermionic Fock spaces parametrized by vector potentials in odd space dimensions and a proof that this leads in a simple manner to the known Schwinger terms (Faddeev-Mickelsson cocycle) for the gauge group action. This gives an explicit computation of the Dixmier-Douady class of the associated bundle gerbe. The method works also in other cases of fermions in external fields (external gravitational field, for example) provided that the APS theorem can be applied; however, we have worked out the details only in the case of vector potentials. Another example arises from the WZW model on Riemann surfaces. A common viewpoint for both of these applications is the ‘lifting’ problem for principal bundles formulated in terms of bundle gerbes. The latter is an abstract version of the ‘existence of string structures’ question.

1. INTRODUCTION

In [Br] Brylinski describes Giraud’s theory of gerbes. Loosely speaking a gerbe over a manifold M is a sheaf of groupoids over M . Gerbes, via their Dixmier-Douady class, provide a geometric realisation of the elements of $H^3(M, \mathbb{Z})$ analogous to the way that line bundles provide, via their Chern class, a geometric realisation of the elements of $H^2(M, \mathbb{Z})$.

There is a simpler way of achieving this end which somewhat surprisingly is nicely adapted to the kind of geometry arising in quantum field theory applications. For want of a better name these objects are called *bundle gerbes* and are introduced in [Mu]. All this talk of sheaves and groupoids sounds very abstract. In this article we want to illustrate the importance and usefulness of bundle gerbes by describing four natural examples arising in different parts of quantum field theory. These are:

- string structures,
- \mathcal{U}_{res} bundles,
- the Wess-Zumino-Witten action, and
- the Faddeev-Mickelsson cocycle.

Just as a gerbe is a sheaf of groupoids a bundle gerbe is essentially a bundle of groupoids. A bundle gerbe has associated to it a three-class also called the Dixmier-Douady class. Every bundle gerbe gives rise to a gerbe with the same Dixmier-Douady class. Bundle gerbes behave in many ways like line bundles. There is a notion of a trivial bundle gerbe and a bundle gerbe is trivial if and only if its Dixmier-Douady class vanishes. One can form the dual and tensor products of bundle gerbes and the Dixmier-Douady class changes sign on the dual and is additive for tensor products. Every bundle gerbe admits a bundle gerbe connection which can be used to define a three form on M called the curvature of the connection

and which is a de Rham representative for the non-torsion part of the Dixmier-Douady class. A difference with the line bundle case is that one needs to choose not just the connection but an intermediate two-form called the curving to define the curvature. There is a notion of holonomy for a connection and curving but now it is associated to a two-surface rather than a loop. There is a local description of bundle gerbes in terms of transition functions and a corresponding Čech definition of the Dixmier-Douady class. Finally bundle gerbes can be pulled back and there is a universal bundle gerbe and an associated classifying theory.

The one significant difference between the two structures; lines bundles and bundle gerbes; is that two line bundles are isomorphic if and only if their Chern classes are equal, whereas two bundle gerbes which are isomorphic have the same Dixmier-Douady class but the converse is not necessarily true. For bundle gerbes there is a weaker notion of isomorphism called *stable isomorphism* and two bundle gerbes are stably isomorphic if and only if they have the same Dixmier-Douady class [MuSt]. The reader with some knowledge of groupoid or category theory will recall that often the right concept of equal for categories is that of equivalence which is weaker than isomorphism. A similar situation arises for bundle gerbes essentially because they are bundles of groupoids.

A common thread in the four examples we consider is the relationship between bundle gerbes and central extensions. Because group actions in quantum field theory are only projectively defined one often needs to consider the so-called ‘lifting problem’ for principal G bundles where G is the quotient in a central extension:

$$U(1) \rightarrow \hat{G} \rightarrow G.$$

The lifting problem starts with a principal G bundle and seeks to find a lift of this to a principal \hat{G} bundle. The obstruction to such a lift is well known to be a class in $H^3(M, \mathbb{Z})$. The connection with bundle gerbes arises because there is a so-called lifting bundle gerbe, which is trivial if and only if the principal G bundle lifts and its Dixmier-Douady class is the three class obstructing the lifting. In the first and third examples G is the loop group with its standard central extension by the Kac-Moody group, in the second example G is the restricted unitary group \mathcal{U}_{res} with its central extension, and in the fourth it is a gauge group. It is important to note, and the central point of this paper, that the bundle gerbe description arises naturally and usefully in these examples and is not just a fancier way of describing the lifting problem for principal bundles.

In summary we present the basic theory of bundle gerbes in Section 2. This is followed by the examples: the restricted unitary bundles and string structures in Section 5, the Faddeev-Mickelsson cocycle in Section 4 and the Wess-Zumino-Witten term in Section 3.

2. BUNDLE GERBES

In this section we describe the basic theory of bundle gerbes. We will not prove any of the results but refer the reader to [Mu, CaMiMu, MuSt, CaMu] and the forthcoming thesis of Stevenson [St].

2.1. The definition and basic operations. Consider a submersion

$$\pi: Y \rightarrow M$$

and define $Y_m = \pi^{-1}(m)$ to be the fibre of Y over $m \in M$. Recall that the fibre product $Y^{[2]}$ is a new submersion over M whose fibre at m is $Y_m \times Y_m$.

A bundle gerbe (P, Y) over M is defined to be a choice of a submersion $Y \rightarrow M$ and a $U(1)$ bundle $P \rightarrow Y^{[2]}$ with a product, that is, a $U(1)$ bundle isomorphism $P \circ P \rightarrow P$ covering $((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3)$. The product is required to be associative whenever triple products are defined.

Example 2.1. Let $Q \rightarrow Y$ be a principal $U(1)$ bundle. Define

$$P_{(x,y)} = \text{Aut}_{U(1)}(Q_x, Q_y) = Q_x^* \otimes Q_y$$

Then this defines a bundle gerbe called the trivial bundle gerbe. We also have $P = \text{Aut}(\pi_1^{-1}Q, \pi_2^{-1}Q) = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$.

A morphism of bundle gerbes (P, Y) over M and (Q, X) over N is a triple of maps (α, β, γ) . The map $\beta: Y \rightarrow X$ is required to be a morphism of the submersions $Y \rightarrow M$ and $X \rightarrow N$ covering $\gamma: M \rightarrow N$. It therefore induces a morphism $\beta^{[2]}$ of the submersions $Y^{[2]} \rightarrow M$ and $X^{[2]} \rightarrow N$. The map α is required to be a morphism of $U(1)$ bundles covering $\beta^{[2]}$ which commutes with the product and hence also with the identity and inverse. A morphism of bundle gerbes over M is a morphism of bundle gerbes for which $M = N$ and γ is the identity on M .

Various constructions are possible with bundle gerbes. We can define a pull-back and product as follows. If (Q, X) is a bundle gerbe over N and $f: M \rightarrow N$ is a map then we can pull back the submersion $X \rightarrow N$ to obtain a submersion $f^{-1}(X) \rightarrow M$ and a morphism of submersions $f^{-1}: f^{-1}(X) \rightarrow X$ covering f . This induces a morphism $(f^{-1}(X))^{[2]} \rightarrow X^{[2]}$ and hence we can use this to pull back the $U(1)$ bundle Q to a $U(1)$ bundle $f^{-1}(Q)$ say on $f^{-1}(X)$. It is easy to check that $(f^{-1}(Q), f^{-1}(X))$ is a bundle gerbe, the pull-back by f of the gerbe (Q, X) . If (P, Y) and (Q, X) are bundle gerbes over M then we can form the fibre product $Y \times_M X \rightarrow M$ and then form a $U(1)$ bundle $P \otimes Q$ over $(Y \times_M X)^{[2]}$ which we call the product of the bundle gerbes (P, Y) and (Q, X) .

Notice that for any $m \in M$ we can define a *groupoid* as follows. The objects of the groupoid are the points in $Y_m = \pi^{-1}(m)$. The morphisms between two objects $x, y \in Y_m$ are the elements of $P_{(x,y)}$. The bundle gerbe product defines the groupoid product. The existence of identity and inverse morphisms is shown in [Mu]. Hence we can think of the bundle gerbe as a family of groupoids, parametrised by M .

2.2. The Dixmier-Douady class and stable isomorphism. Let $\{U_\alpha\}$ be an open cover of M such that over each U_α we can find sections $s_\alpha: U_\alpha \rightarrow Y$. Then over intersections $U_\alpha \cap U_\beta$ we can define a map

$$(s_\alpha, s_\beta): U_\alpha \cap U_\beta \rightarrow Y^{[2]}$$

which sends m to $(s_\alpha(m), s_\beta(m))$. If we choose a sufficiently nice cover we can then find maps

$$\sigma_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow P$$

such that $\sigma_{\alpha\beta}(m) \in P_{(s_\alpha(m), s_\beta(m))}$. The $\sigma_{\alpha\beta}$ are sections of the pull-back of P by (s_α, s_β) . By using the bundle gerbe multiplication (written here as juxtaposition) we have that

$$\sigma_{\alpha\beta}(m)\sigma_{\beta\gamma}(m) \in P_{(s_\alpha(m), s_\gamma(m))}$$

and hence can be compared to $\sigma_{\alpha\gamma}(m)$. The difference is an element of $U(1)$ defined by

$$\sigma_{\alpha\beta}(m)\sigma_{\beta\gamma}(m) = \sigma_{\alpha\gamma}(m)g_{\alpha\beta\gamma}(m)$$

and this defines a map

$$g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1).$$

It is straightforward to check that this is a cocycle and defines an element of $H^2(M, \underline{U(1)})$ independent of all the choices we have made. Here, if G is a Lie group we use the notation \underline{G} for the sheaf of smooth maps into G . It is a standard result that the coboundary map

$$\delta: H^2(M, \underline{U(1)}) \rightarrow H^3(M, \mathbb{Z}) \quad (2.1)$$

induced by the short exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \rightarrow \underline{\mathbb{C}} \rightarrow \underline{U(1)} \rightarrow 0$$

is an isomorphism. Either the class defined by $g_{\alpha\beta\gamma}$ or its image under the coboundary map is called the Dixmier-Douady class of the bundle gerbe. We denote it by $d(Q, Y)$.

The first important fact about the Dixmier-Douady class is

Proposition 2.1 ([Mu]). *A bundle gerbe is trivial if and only its Dixmier-Douady class is zero.*

Let (P, Y) and (Q, X) be bundle gerbes over M and $Z \rightarrow M$ be a map admitting local sections with $f: Z \rightarrow Y$ a map commuting with projections to M . Then from [Mu] we have

Theorem 2.1 ([Mu]). *If P and Q are bundle gerbes over M then*

- (1) $d(P^*, Y) = -d(P, Y)$
- (2) $d(P \otimes Q, Y \times_M X) = d(P, Y) + d(Q, X)$, and
- (3) $d(f^*(P), X) = f^*(d(P, Y))$

Because pulling back the submersion does not change the Dixmier-Douady class of a bundle gerbe it is clear there are many non-isomorphic bundle gerbes with the same Dixmier-Douady class. We define

Definition 2.1. Two bundle gerbes (P, Y) and (Q, Z) are defined to be stably isomorphic if there are trivial bundle gerbes T_1 and T_2 such that

$$P \otimes T_1 = Q \otimes T_2.$$

We have the following theorem:

Theorem 2.2 ([Mu]). *For bundle gerbes (P, Y) and (Q, Z) the following are equivalent.*

- (1) P and Q are stably isomorphic
- (2) $P \otimes Q^*$ is trivial
- (3) $d(P) = d(Q)$.

2.3. Local bundle gerbes. The notion of stable isomorphism is useful in understanding the role of open covers in bundle gerbes. Let (P, Y) be a bundle gerbe and assume we have an open cover and various maps s_α , $\sigma_{\alpha\beta}$ and $g_{\alpha\beta\gamma}$ as defined in subsection 2.2. Let X be the disjoint union of all the open sets U_α . This can be thought of as all pairs (α, m) where $m \in U_\alpha$. There is a projection $X \rightarrow M$ defined by $(\alpha, m) \mapsto m$ which admits local sections. Moreover there is a map $s: X \rightarrow Y$ preserving projections defined by $s(\alpha, m) = s_\alpha(m)$.

The pullback by s of the bundle gerbe (P, Y) is stably isomorphic to (P, Y) [MuSt]. This pull-back consists of a collection of $U(1)$ bundles $Q_{\alpha\beta} \rightarrow U_\alpha \cap U_\beta$. On triple overlaps there is a bundle map

$$Q_{\alpha\beta} \otimes Q_{\beta\gamma} \rightarrow Q_{\alpha\gamma}$$

which on quadruple overlaps is associative in the appropriate sense. A completely local description of bundle gerbes can be given in terms of open covers, $U(1)$ bundles on double overlaps and products on triple overlaps [St]. The results on stable isomorphism tell us that this is equivalent to working with bundle gerbes.

2.4. Bundle gerbe connections, curving and curvature. Because $P \rightarrow Y^{[2]}$ is a $U(1)$ bundle it has connections. It is shown in [Mu] that it admits *bundle gerbe connections* that is connections commuting with the bundle gerbe product. It is also shown there that the curvature F of such a connection must satisfy the ‘descent equation’:

$$F = \pi_1^*(f) - \pi_2^*(f)$$

for a two-form f on Y . The two-form f is not unique and a choice of an f is called a *curving* for the bundle gerbe connection. It is then easy to show that $df = \pi^*(\omega)$ for some three-form ω on M . In [Mu] it is shown that ω is a de Rham representative for the image of the Dixmier-Douady class in real cohomology.

2.5. The lifting bundle gerbe. Finally let us conclude this section with the motivating example of the so-called lifting bundle gerbe. That is the bundle gerbe arising from the lifting problem for principal bundles. Consider a central extension of groups:

$$1 \rightarrow U(1) \xrightarrow{\iota} \hat{G} \xrightarrow{p} G \rightarrow 1 \tag{2.2}$$

and a principal G bundle $Y \rightarrow M$. Then it may happen that there is a principal \hat{G} bundle \hat{Y} and a bundle map $\hat{Y} \rightarrow Y$ commuting with the homomorphism $\hat{G} \rightarrow G$. In such a case Y is said to lift to a \hat{G} bundle. One way of answering the question of when Y lifts to a \hat{G} bundle is to present Y with transition functions $g_{\alpha\beta}$ relative to a cover $\{U_\alpha\}$ of M . If the cover is sufficiently nice we can lift the $g_{\alpha\beta}$ to maps $\hat{g}_{\alpha\beta}$ taking values in \hat{G} and such that $p(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. These are candidate transition functions for a lifted bundle \hat{Y} . However they may not satisfy the cocycle condition $\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = 1$ and indeed there is a $U(1)$ valued map

$$e_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1)$$

defined by $\iota(e_{\alpha\beta\gamma}) = \hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha}$. Because (2.2) is a central extension it follows that $e_{\alpha\beta\gamma}$ is a cocycle and hence defines a class in $H^2(M, U(1))$. Using the isomorphism (2.1) we define a class in $H^3(M, \mathbb{Z})$ which is the obstruction to solving the lifting problem for P .

Consider the fibre product of P with itself. There is a map $g: P^{[2]} \rightarrow G$ defined by $p_1 = p_2g(p_1, p_2)$ and we can use this to pull back the $U(1)$ bundle $\hat{G} \rightarrow G$ to define a $U(1)$ bundle $Q \rightarrow P^{[2]}$. More concretely we have

$$Q_{(p_1, p_2)} = \{g \in \hat{G} \mid p_1p(g) = p_2\}.$$

The group product on \hat{G} induces a bundle gerbe product on Q . It is shown in [Mu] that the bundle gerbe Q is trivial if and only if the bundle P lifts to \hat{G} and moreover the Dixmier-Douady class of (Q, P) is the class defined in the preceding paragraph.

3. THE WESS-ZUMINO-WITTEN TERM

In quantum field theory the path integral can have contributions that are topological in nature. Often these arise as the *holonomy* of a connection. For example if $L \rightarrow M$ is a hermitian line bundle we can consider the Hilbert space of all L^2 sections of L as a space of states. A connection ∇ on L defines an operator on states by

$$K_{\nabla}(\psi)(x) = \int_M \int_{\gamma \in W_{x,y}} P_{\gamma}(\nabla)(\psi(y)) dM$$

where $W_{x,y}$ is the set of all paths from x to y and

$$P_{\gamma}(\nabla): L_y \rightarrow L_x$$

is the operation of parallel transport along the curve γ from the fibre of L over y to the fibre of L over x . If $x = y$ then $P_{\gamma}(\nabla): L_y \rightarrow L_x$ is an element of $U(1)$ called $\text{hol}(\gamma, \nabla)$, the *holonomy* of the connection ∇ around the loop gamma. Because S^2 is simply connected every loop γ bounds a disk D and we have the fact that

$$\text{hol}(\gamma, \nabla) = \exp\left(\int_D F_{\nabla}\right) \quad (3.1)$$

where F_{∇} , is the curvature two-form of ∇ .

The Wess-Zumino-Witten term is defined as follows. The space of states is the space of all maps ψ from a closed Riemann surface Σ into a compact Lie group G . Let X be a three-manifold whose boundary is Σ and $\psi: \Sigma \rightarrow G$. Then ψ can be extended to a map $\hat{\psi}: X \rightarrow G$. Let ω be a closed three-form on G such that $\omega/2\pi i$ generates the integral cohomology of G . The Wess-Zumino-Witten action of f is then defined by

$$\text{WZW}(f) = \exp \int_X \hat{\psi}^*(\omega).$$

It follows from the integrality of $\omega/2\pi i$ that this is independent of the choice of extension $\hat{\psi}$. Clearly this is analogous to defining the holonomy by using the right hand side of equation (3.1) as if one knew nothing of connections, only that F was a two-form such that $F/2\pi i$ was integral.

It is natural to look for the analogous left hand side of equation (3.1) in the case of the Wess-Zumino-Witten term. In [CaMu2] the first and last named authors gave an interpretation of the Wess-Zumino-Witten term in terms of holonomy of a connection on a line bundle over the loop group of G . This essentially only worked for simple Riemann surfaces such as spheres or cylinders. In [Ga] Gawedski gave a construction that works for any Riemann surface. Gawedski starts by showing that isomorphism classes of line bundles with connection are classified by certain Deligne cohomology groups which can be realised in terms of Čech cohomology of

an open cover of M . It is then shown that if M is a loop then this cohomology group is $U(1)$ and the identification of isomorphism classes with elements of $U(1)$ is just the holonomy of the connection around the loop. This is then generalised to the Wess-Zumino-Witten case. For such cohomology classes Gawedski shows that it is possible to define a holonomy associated to a closed Riemann surface in M .

Bundle gerbes give a geometric interpretation of Gawedski's results. The Deligne cohomology class in question defines a stable isomorphism class of a bundle gerbe with connection and curving over M and the element of $U(1)$ is the holonomy of the connection and curving over the surface Σ . In the case that $Y \rightarrow M$ admits local sections a definition of the holonomy in terms of lifting Σ to Y is given in [Mu]. In the present work we are interested in more general Y , in particular a Y arising from an open cover, so we give an alternative definition of holonomy of a bundle gerbe connection and curving.

To define the holonomy we need the notion of a stable isomorphism class of a bundle gerbes with connection and curving. To define this let $P \rightarrow Y$ be a line bundle with connection ∇ and curvature F . The trivial gerbe $\delta(P) \rightarrow Y^{[2]}$ has a natural bundle gerbe connection $\pi_1^*(\nabla) = \pi_2^*(\nabla)$ and curving $\pi_1^*(F) = \pi_2^*(F)$. To extend the definition of stable isomorphism (definition 2.1) to cover the case of bundle gerbes with connection and curving we assume that the trivial bundle gerbes T_1 and T_2 , in the definition, have connections arising in this manner and that the isomorphism in definition 2.1 preserves connections. Then stable isomorphism classes of bundle gerbes with connection and curving are classified by the two dimensional Deligne cohomology or the hyper-cohomology of the log complex of sheaves

$$0 \rightarrow \Omega^\times \xrightarrow{d^{\log}} \Omega^1 \rightarrow \Omega^2 \rightarrow 0.$$

The hypercohomology of this sequence of sheaves can be calculated by taking a Leray cover $\mathcal{U} = \{U_\alpha\}$ of M and considering the double complex:

$$\begin{array}{ccccccc} \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow \\ C^2(\mathcal{U}, \Omega^\times) & \xrightarrow{d^{\log}} & C^2(\mathcal{U}, \Omega^1) & \rightarrow & C^2(\mathcal{U}, \Omega^2) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^3) & \xrightarrow{d} \\ \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & \\ C^1(\mathcal{U}, \Omega^\times) & \xrightarrow{d^{\log}} & C^1(\mathcal{U}, \Omega^1) & \rightarrow & C^1(\mathcal{U}, \Omega^2) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^3) & \xrightarrow{d} \\ \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & \\ C^0(\mathcal{U}, \Omega^\times) & \xrightarrow{d^{\log}} & C^0(\mathcal{U}, \Omega^1) & \rightarrow & C^0(\mathcal{U}, \Omega^2) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^3) & \xrightarrow{d} \end{array} \quad (3.2)$$

The Deligne cohomology is the cohomology of the double complex 3.2. This can be calculated by either of two spectral sequences which begin by taking the cohomology along the rows or columns respectively. If we assume that M is a surface Σ and take the cohomology of the columns we obtain the E^2 term of the spectral sequence:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ H^1(\Sigma, \Omega^\times) & 0 & 0 & 0 \\ H^0(\Sigma, \Omega^\times) & H^1(\Sigma, \mathbb{C}) & H^2(\Sigma, \mathbb{C}) & H^3(\Sigma, \mathbb{C}) \end{array}$$

where we use the fact that the sheaves Ω^i have no cohomology and that $H^2(\Sigma, \Omega^\times) = H^3(\Sigma, \mathbb{Z}) = 0$ because Ω^\times is the sheaf $U(1)$.

The third cohomology is therefore the quotient of the image of the inclusion

$$\mathbb{Z} = H^2(\Sigma, \mathbb{Z}) = H^1(\Sigma, \Omega^\times) \rightarrow H^2(\Sigma, \mathbb{C}) = \mathbb{C}$$

induced by the second differential. It is straightforward to check that this map is just the natural inclusion $\mathbb{Z} \rightarrow \mathbb{C}$ and hence the quotient is just $U(1)$. We call the resulting non-vanishing number attached to each connection and curving the holonomy, $\text{hol}(\Sigma, \nabla, f)$, where ∇ is the bundle gerbe connection and f is the curving. If $\psi: \Sigma \rightarrow M$ is a map then we can pull-back the bundle gerbe, connection and curving to Σ and we define $\text{hol}(\psi, \nabla, f)$ to be the holonomy of the pulled-back connection and curving over Σ .

To calculate the holonomy we need to explain how to unravel these definitions. Let us begin with a bundle gerbe (Q, Y) and choose a Leray cover $\mathcal{U} = \{U_\alpha\}$ with sections $s_\alpha: U_\alpha \rightarrow Q$. Choose sections $\sigma_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow Q$ as we did in subsection 2.2. and define $g_{\alpha\beta\gamma}$ by

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma}^{-1}\sigma_{\gamma\alpha} = g_{\alpha\beta\gamma}.$$

Let $A_{\alpha\beta}$ be the pullback of the connection one form on Q by $\sigma_{\alpha\beta}$ and f_α the pullback of the curving by s_α . These satisfy

$$\begin{aligned} d \log g_{\alpha\beta\gamma} &= A_{\alpha\beta} - A_{\beta\gamma} + A_{\gamma\alpha} \\ dA_{\alpha\beta} &= f_\alpha - f_\beta \end{aligned}$$

and hence $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, f_\alpha)$ is an element of the total cohomology of the complex (3.2). Because Σ is two dimensional the cocycle $g_{\alpha\beta\gamma}$ is trivial and we can solve the equation

$$g_{\alpha\beta\gamma} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta}$$

where $h_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow U(1)$ and hence

$$d \log h_{\alpha\beta} = A_{\alpha\beta} + k_\alpha - k_\beta$$

where the k_α are one-forms on U_α . Hence the two-form $f_\alpha - dk_\alpha$ on U_α agrees with the two-form $f_\beta - dk_\beta$ on U_β on the overlap $U_\alpha \cap U_\beta$ and hence defines a two-form on Σ . The exponential of 2π times this two-form over Σ is the holonomy of the connection and curving.

It is straightforward to check that if we can extend the map of Σ into M to a map of a manifold X whose boundary is Σ then we obtain the analogue of (3.1)

$$\text{hol}(\nabla, f, \psi) = \exp\left(\int_X \psi^*(\omega)\right)$$

where ω is three curvature of the connection and curving.

In the case that Σ has boundary one expects a result analogous to parallel transport along a curve γ . Gawedski shows that there is a naturally defined line bundle L over the space LM of loops in M . The boundary components b_1, \dots, b_r of Σ define points in LM and Gawedski shows that the holonomy can be interpreted as an element of

$$L_{b_1} \otimes L_{b_2} \otimes \cdots \otimes L_{b_r}$$

for more details see [Ga].

4. THE FADDEEV-MICKELSSON COCYCLE

We begin with a general observation. First let (J, Y) be a bundle gerbe over M and assume that M has an open cover $\{U_\alpha\}_{\alpha \in I}$ such that there are sections $s_\alpha: U_\alpha \rightarrow Y$ for each α . Consider the disjoint union $\coprod_{\alpha \in I} U_\alpha$. It is useful to think

of this as a subset of $M \times I$. Then the projection map to M restricts to define a submersion. The sections s_α define a map

$$s: \coprod_{\alpha \in I} U_\alpha \rightarrow Y$$

by $s(x, \alpha) = s_\alpha(x)$. This commutes with the projections and hence we can pull-back the bundle gerbe J to a bundle gerbe on the disjoint union. From the discussion above this is stably isomorphic to the original bundle gerbe.

Next let M be a smooth compact manifold without boundary equipped with a spin structure. We assume that the dimension of M is odd and equal to $2n + 1$. Let S be the spin bundle over M , with fiber isomorphic to \mathbb{C}^{2^n} . Let H be the space of square integrable sections of the complex vector bundle $S \otimes V$, where V is a trivial vector bundle over M with fiber to be denoted by the same symbol V . The measure is defined by a fixed metric on M and V . We assume that a unitary representation ρ of a compact group G is given in the fiber. The set of smooth vector potentials on M with values in the Lie algebra \mathfrak{g} of G is denoted by \mathcal{A} . The topology on \mathcal{A} arises from an infinite family of Sobolev norms which via the Sobolev embedding theorem give a metric equivalent to that arising from the topology of uniform convergence of derivatives of all orders.

For each $A \in \mathcal{A}$ there is a massless hermitean Dirac operator D_A . Fix a potential A_0 such that D_{A_0} does not have the zero as an eigenvalue and let H_+ be the closed subspace spanned by eigenvectors belonging to positive eigenvalues of D_{A_0} and H_- its orthogonal complement (with corresponding spectral projections P_\pm). More generally for any potential A and any real λ not belonging to the spectrum of D_A we define the spectral decomposition $H = H_+(A, \lambda) \oplus H_-(A, \lambda)$ with respect to the operator $D_A - \lambda$. Let \mathcal{A}_0 denote the set of all pairs (A, λ) as above and let

$$U_\lambda = \{A \in \mathcal{A} | (A, \lambda) \in \mathcal{A}_0\}.$$

Over the set $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ there is a canonical complex line bundle, which we denote by $DET_{\lambda\lambda'}$. Its fiber at $A \in U_{\lambda\lambda'}$ is the top exterior power

$$DET_{\lambda\lambda'}(A) = \wedge^{\text{top}}(H_+(A, \lambda) \cap H_-(A, \lambda'))$$

where we have assumed $\lambda < \lambda'$. For completeness we put $DET_{\lambda\lambda'} = DET_{\lambda'\lambda}^{-1}$. Since M is compact, the spectral subspace corresponding to the interval $[\lambda, \lambda']$ in the spectrum is finite-dimensional and the complex line above is well-defined.

It is known [Mi1, CaMu1] that there exists a complex line bundle DET_λ over each of the sets U_λ such that

$$DET_{\lambda'} = DET_\lambda \otimes DET_{\lambda\lambda'} \tag{4.1}$$

over the set $U_{\lambda\lambda'}$. In [CaMu, CaMu1] the structure of these line bundles was studied with the help of bundle gerbes. In particular, there is an obstruction for passing to the quotient by the group \mathcal{G} of gauge transformations which is given by the Dixmier-Douady class of the bundle gerbe. (In [Mi1] the structure of the bundles and their relation to anomalies was found by using certain embeddings to infinite-dimensional Grassmannians.)

We shall describe the computation in [CaMiMu] of the curvature of the (odd dimensional) determinant bundles from Atiyah-Patodi-Singer index theory and how to obtain the Schwinger terms in the Fock bundle directly from the local part of the index density.

We may consider \mathcal{A}_0 as part of a bundle gerbe over \mathcal{A} . The obvious map $\mathcal{A}_0 \rightarrow \mathcal{A}$ is a submersion. For any $\lambda \in \mathbb{R}$ we have a section $s_\lambda: U_\lambda \rightarrow \mathcal{A}_0$ defined by $s_\lambda(A) = (A, \lambda)$. So we can apply the discussion above to obtain the disjoint union

$$Y = \coprod U_\lambda \subset \mathcal{A} \times \mathbb{R}$$

as the set of all (A, λ) such that $A \in U_\lambda$. We topologize Y by giving \mathbb{R} the discrete topology. Notice that as a set Y is just \mathcal{A}_0 but the topology is different. The identity map $Y \rightarrow \mathcal{A}_0$ is continuous. So from the result above we deduce that bundle gerbes over \mathcal{A}_0 with either topology on \mathcal{A}_0 are stably isomorphic so we can work with either picture. An advantage of the open cover picture is that the map δ introduced in [Mu] is then just the coboundary map in the C ech de-Rham double complex. In the next section \mathcal{A}_0 can be interpreted in either sense.

If we restrict λ to be rational then the sets U_λ form a denumerable cover. It follows that the intersections $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ also form a denumerable open cover. Similarly, we have an open cover by sets $V_{\lambda\lambda'} = \pi(U_{\lambda\lambda'})$ on the quotient $\mathcal{X} = \mathcal{A}/\mathcal{G}_e$, where \mathcal{G}_e is the group of *based* gauge transformations g , $g(p) = e$ the identity at some fixed base point $p \in M$. Here $\pi: \mathcal{A} \rightarrow \mathcal{X}$ is the canonical projection.

We now describe the bundle gerbe J over \mathcal{A} defined in [CaMu] and extracted from the work of [Se]. This is a line bundle over the fibre product $\mathcal{A}_0^{[2]}$. This fibre product can be identified with all triples (A, λ, λ') where neither λ nor λ' are in the spectrum of D_A . The fibre of J over (A, λ, λ') is just $DET_{\lambda\lambda'}$. For this to be a bundle gerbe we need a product which in this case is a linear isomorphism

$$DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''} = DET_{\lambda\lambda''}.$$

But such a linear isomorphism is a simple consequence of the definition of $DET_{\lambda\lambda'}$ and the fact that taking highest exterior powers is multiplicative for direct sums.

Let $\pi: \mathcal{A}_0 \rightarrow \mathcal{A}$ be the projection. Let $p: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}_1$ be the quotient by the gauge action. We saw in [CaMu1] that the line bundle DET on \mathcal{A}_0 satisfies $J = \delta(DET)$. Here $\delta(DET) = \pi_1^*(DET)^* \otimes \pi_2^*(DET)$ where $\pi_i: \mathcal{A}_0^{[2]} \rightarrow \mathcal{A}_0$ are the projections,

$$\pi_1((A, \lambda, \lambda')) = (A, \lambda), \quad \pi_2((A, \lambda, \lambda')) = (A, \lambda').$$

In other words $J = \delta(DET)$ is equivalent to

$$DET_{\lambda\lambda'} = DET_\lambda^* \otimes DET_{\lambda'}$$

which is equivalent to equation 4.1.

The fibering $\mathcal{A}_0 \rightarrow \mathcal{A}$ has, over each open set U_λ a canonical section $A \mapsto (A, \lambda)$. These enable us to suppress the geometry of the submersion and the bundle gerbe J becomes the line bundle $DET_{\lambda\lambda'}$ over the intersection $U_{\lambda\lambda'}$ and its triviality amounts to the fact that we have the line bundle DET_λ over U_λ and over intersections we have the identifications

$$DET_{\lambda\lambda'} = DET_\lambda^* \otimes DET_{\lambda'}.$$

We denote the Chern class of $DET_{\lambda\lambda'}$ by $\theta_2^{[2]}$. Note that these bundles descend to bundles over $V_{\lambda\lambda'} = \pi(U_{\lambda\lambda'}) \subset \mathcal{A}/\mathcal{G}_e$. Therefore, the forms $\theta_2^{\lambda\lambda'} = \theta_2^\lambda - \theta_2^{\lambda'}$ on $U_{\lambda\lambda'}$ (where θ_2^λ is the 2-form giving the curvature of DET_λ) are equivalent (in cohomology) to forms which descend to closed 2-forms $\phi_2^{\lambda\lambda'}$ on $V_{\lambda\lambda'}$. The following result is established in [CaMiMu]

Theorem 4.1. [CaMiMu] *The family of closed 2-forms $\phi_2^{\lambda\lambda'}$ on $V_{\lambda\lambda'}$ determines a representative for the Dixmier-Douady class ω of the bundle gerbe J/\mathcal{G}_e . In addition, noting that $\delta(DET) = J$, the connection with the Faddeev-Mickelsson cocycle on the Lie algebra of the gauge group is simply that it is cohomologous to the negative of the cocycle defined by the curvature F_{DET} of the line bundle DET .*

To obtain the Dixmier-Douady class as a characteristic class we recall that in the case of even dimensional manifolds, Atiyah and Singer [AtSi] gave a construction of ‘anomalies’ in terms of characteristic classes. In the present case of odd dimensional manifolds a similar procedure yields the Dixmier-Douady class.

We begin with the observation that given a closed integral form Ω of degree p on a product manifold $M \times X$ ($\dim M = d$ and $\dim X = k$) we obtain a closed integral form on X , of degree $p - d$, as

$$\Omega_X = \int_M \Omega.$$

If now A is any Lie algebra valued connection on the product $M \times X$ and F is the corresponding curvature we can construct the Chern form $c_{2n} = c_{2n}(F)$ as a polynomial in F . Apply this to the connection A defined by Atiyah and Singer, [AtSi], [DoKr p. 196], in the case when $X = \mathcal{A}/\mathcal{G}_e$.

First pull back the forms to $M \times \mathcal{A}$. The Atiyah-Singer connection on $M \times X$ becomes a globally defined Lie algebra valued 1-form \hat{A} on $M \times \mathcal{A}$. Let \hat{F} be the curvature form determined by \hat{A} . We showed in [CaMiMu] that

$$\int_{S^3} \Omega_X = \int_{M \times D^3} c_{2n}(\hat{F}).$$

where D^3 is the pullback to \mathcal{A} of S^3 . But the integral of the Chern form over a manifold with a boundary (when the potential is globally defined) is equal to the integral of the Chern-Simons action:

$$\int_{M \times \partial D^3} CS_{2n-1}(\hat{A}).$$

Along gauge directions the form \hat{A} is particularly simple so for example when $M = S^1$ and $2n = 4$ we get (here $S^2 = \partial D^3$)

$$\int_{S^3} \Omega_X = \int CS_3(\hat{A}) = \frac{1}{24\pi^2} \int_{S^1 \times S^2} \text{tr}(dgg^{-1})^3$$

where $g = g(x, z)$ is a family of gauge transformations parametrized by $z \in S^2$. Similar results hold in higher dimensions.

Theorem 4.2. [CaMiMu] *The class Ω_X is a representative for the Dixmier-Douady class of the bundle gerbe J/\mathcal{G}_e .*

5. \mathcal{U}_{res} BUNDLES AND STRING STRUCTURES

Let $H = H_+ \oplus H_-$ be a polarization of a Hilbert space H into a pair of closed infinite-dimensional subspaces. We denote by \mathcal{U}_{res} the restricted unitary group consisting of unitary operators in H such that the off-diagonal blocks are Hilbert-Schmidt operators. In a recent preprint [CaCrMu] we described in some detail results about the Dixmier-Douady class arising from the problem of lifting principal \mathcal{U}_{res} bundles to principal $\hat{\mathcal{U}}_{res}$ bundles. Here $\hat{\mathcal{U}}_{res}$ is a central extension of \mathcal{U}_{res} , [PrSe]. We now summarise the results in [CaCrMu].

Theorem 5.1. *There is an imbedding of the smooth loop group LG of a compact Lie group G in \mathcal{U}_{res} which extends to give an imbedding of the canonical central extension \widehat{LG} in $\widehat{\mathcal{U}}_{res}$. Under this imbedding the obstruction to the existence of a string structure (in the sense of Killingback [Kil]: a \widehat{LG} principal bundle) on the loop space of a manifold M may be identified with the Dixmier-Douady class of the bundle gerbe of the corresponding principal \mathcal{U}_{res} bundle.*

A different approach to the question of the existence of string structures is due to [Mc] and exploits Brylinski's point of view whereas in [CaMu] the problem is solved using the classifying map of the bundle over the loop space of M .

There is also an imbedding of \mathcal{U}_{res} in the projective unitary group of the skew symmetric Fock space (determined by the polarization $H = H_+ \oplus H_-$, the 'Dirac sea' construction), [PrSe]. Under conditions on the underlying manifold M this imbedding enables us to establish a relationship between the Dixmier-Douady class of a bundle gerbe over M determined by a \mathcal{U}_{res} bundle and the second Chern class of an associated principal projective unitary group bundle over the suspension of M .

In this section we describe the field theory examples which motivated the proving of the previous results.

Let $\mathcal{G}r$ be the space of all closed infinite-dimensional subspaces of H with the topology determined by operator norm topology for the associated projections. We may think of $\mathcal{G}r$ as the homogeneous space

$$U(H)/(U(H_+) \times U(H_-)).$$

Here all the groups are contractible (in the operator norm topology) and therefore there is a continuous section $\mathcal{G}r \rightarrow U(H)$, that is, for $W \in \mathcal{G}r$ we may choose a $g_W \in U(H)$ which depends continuously on W , such that $W = g_W \cdot H_+$.

The example we shall study below comes from a quantization of a family of Dirac operators D_A parametrized by smooth (static) vector potentials A . In the following we shall use the notations in section 3.

Choose a real number λ such that $D = D_A - \lambda$ is invertible. The set of bounded operators X such that $\|X\| < \|1/D\|^{-1}$ is an open set V containing 0 and the function $X \mapsto |D + X|^{-1}(D_0 + X)$ is continuous in the operator norm of X ; this is seen using the geometric converging geometric series $(D + X)^{-1} = \frac{1}{D} - \frac{1}{D}X\frac{1}{D} + \dots$. Since the operator norm of the interaction A depends continuously on the components A_i of the vector potential (with respect to the infinite family of Sobolev norms on \mathcal{A}) we can conclude that $A \mapsto \epsilon_{A,\lambda} = (D_A - \lambda)/|D_A - \lambda|$ is continuous. Thus also the spectral projections $P_{\pm}(A, \lambda) = \frac{1}{2}(1 \pm \epsilon_{A,\lambda})$ are continuous and $H_+(A, \lambda) = P_+(A, \lambda)H \in \mathcal{G}r$ depends continuously on $A \in U_{\lambda}$. On the other hand, we know that there is a section $\mathcal{G}r \rightarrow U(H)$ and therefore we may choose a continuous function $A \mapsto g_{\lambda}(A) \in U(H)$ such that $H_+(A, \lambda) = g_{\lambda}(A) \cdot H_+$. We shall show that these define transition functions, $g_{\lambda\lambda'}(A) = g_{\lambda}(A)^{-1}g_{\lambda'}(A)$, for a principal \mathcal{U}_{res} bundle P over \mathcal{A} . By construction, these satisfy the cocycle property required for transition functions so the only thing which remains is to prove continuity with respect to the topology of \mathcal{U}_{res} .

The topology of \mathcal{U}_{res} is defined by the operator norm topology on the diagonal blocks (with respect to the energy polarization $H_+ \oplus H_-$ fixed by the free Dirac operator D_0) and by Hilbert-Schmidt norm topology on the off-diagonal blocks. Denote $P_{\pm} = P_{\pm}(A_0, 0)$ and $\epsilon = P_+ - P_-$. We already know that the $g_{\lambda\lambda'}$'s (assume

e.g. that $\lambda < \lambda'$) are continuous with respect to the operator norm topology and we need only show that the off-diagonal blocks $[\epsilon, g_{\lambda\lambda'}]$ are continuous in the Hilbert-Schmidt topology. Let us concentrate on the upper right block $K_{+-} = P_+ g_{\lambda\lambda'} P_-$.

Multiplying from the left by g_λ and from the right by $g_{\lambda'}^{-1}$ and using the fact that Hilbert-Schmidt operators form an operator ideal with $\|gK\|_2 \leq \|g\| \cdot \|K\|_2$ we conclude that K_{+-} is continuous in the Hilbert-Schmidt norm if and only if

$$g_\lambda \pi_+ g_\lambda^{-1} g_{\lambda'} \pi_- g_{\lambda'}^{-1}$$

is a continuous function of A in the Hilbert-Schmidt norm. Now the product of the first three factors in the above expression gives $P_+(A, \lambda)$ whereas the product of the last three factors is $P_-(A, \lambda')$. But $P_+(A, \lambda)P_-(A, \lambda')$ is the spectral projection $P(\lambda, \lambda')$ to the finite-dimensional spectral subspace corresponding to the interval $[\lambda, \lambda']$. On the other hand, the dimension of this subspace is fixed over $U_{\lambda\lambda'}$ and therefore the Hilbert-Schmidt norm of the projection, which is the square root of its rank, is continuous. Furthermore, since $P(\lambda, \lambda')$ is continuous in the operator norm and it has a fixed finite rank it is also continuous in the Hilbert-Schmidt norm.

We denote by $\mathcal{G}r_{res}$ the restricted Grassmannian, defined as the orbit $\mathcal{U}_{res} \cdot H_+$ in $\mathcal{G}r$. The fiber P_A at $A \in \mathcal{A}$ can be thought of as the set of all unitary operators $T: H \rightarrow H$ such that $T^{-1}(H_+(A, \lambda))$ (for any λ) is in $\mathcal{G}r_{res}$. This is because $g_\lambda(A)$ provides such an operator for any $A \in U_\lambda$ and any two such operators differ only by a right multiplication by an element of \mathcal{U}_{res} .

Being a principal bundle over a contractible parameter space, $P \rightarrow \mathcal{A}$ is trivial.

On any U_λ the function $A \rightarrow P_+(A, \lambda)$ is continuous and

$$T_A^{-1} P_+(A, \lambda) T_A \in \mathcal{G}r_{res}.$$

Over $\mathcal{G}r_{res}$ there is a canonical determinant bundle DET_{res} . The action of \mathcal{U}_{res} on $\mathcal{G}r_{res}$ lifts to an action of $\hat{\mathcal{U}}_{res}$ on DET_{res} , [PrSe].

Using these maps $A \rightarrow P_+(A, \lambda)$ we can pull back the determinant bundle DET_{res} over $\mathcal{G}r_{res}$ to form local determinant bundles DET_λ over U_λ . This family is the right one for discussing the gerbes over \mathcal{A} and $\mathcal{A}/\mathcal{G}_e$. The reason is that the class of the bundle gerbe is completely determined by the line bundles $DET_{\lambda\lambda'}$ over $U_{\lambda\lambda'}$.

On the restricted Grassmannian we obtain an isomorphism between the fibers $DET_{res}(W)$ and $DET_{res}(W')$, where $W' \subset W$ are points in $\mathcal{G}r_{res}$; the isomorphism is determined by a choice of basis $\{v_1, \dots, v_n\}$ in $W \cap W'^\perp$ as follows. Recalling from [PrSe] that an element in $DET_{res}(W)$ is represented by the so-called admissible basis $\{w_1, w_2, \dots\}$, modulo unitary rotations with determinant equal to one, the isomorphism is simply $\{w_1, w_2, \dots\} \mapsto \{v_1, \dots, v_n, w_1, w_2, \dots\}$. In particular, we apply this when W, W' are the points obtained by mapping $H_+(A, \lambda)$ and $H_+(A, \lambda')$ to $\mathcal{G}r_{res}$ using T_A . Now the vectors $T_A v_i$ span a basis in the subspace corresponding to the interval $[\lambda, \lambda']$ in the spectrum of D_A and thus they define an element in $DET_{\lambda\lambda'}$ in our earlier construction and the basis can be viewed as an isomorphism between DET_λ and $DET_{\lambda'}$.

Next we consider the trivial bundles $\mathcal{A} \times \mathcal{U}_{res}$ and $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ over \mathcal{A} . The gauge group \mathcal{G} acts in the former as follows. Define $\omega(g; A) = T_{g \cdot A}^{-1} g T_A$. This function takes values in \mathcal{U}_{res} and is a 1-cocycle by construction,

$$\omega(gg'; A) = \omega(g; g' \cdot A) \omega(g'; A).$$

Thus the gauge group acts through $g \cdot (A, S) = (g \cdot A, \omega(g; A)S)$ in $\mathcal{A} \times \mathcal{U}_{res}$.

Since ω takes values in \mathcal{U}_{res} the same construction which gives the lifting of the \mathcal{U}_{res} action on \mathcal{G}_{res} to a $\hat{\mathcal{U}}_{res}$ action on DET_{res} gives also an action of an extension $\hat{\mathcal{G}}$ in $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ and in $\mathcal{A} \times DET_{res}$. The pull-back with respect to the conjugation by T_A 's of the latter action defines an action of $\hat{\mathcal{G}}$ on the local determinant bundles DET_λ . Next we observe that the natural action (without center) $v_i \mapsto gv_i$ in the line $DET_{\lambda\lambda'}$ intertwines between the action of the group extension in the lines $DET_\lambda, DET_{\lambda'}$ parametrized by potentials $g \cdot A$ on the gauge orbit. This follows from the corresponding property of the determinant bundle over \mathcal{G}_{res} (by pushing forward by T_A): An element $\hat{g} \in \hat{\mathcal{U}}_{res}$ acts on $w = \{w_1, w_2, \dots\} \in DET_{res}(W)$ as $w_i \mapsto \sum_j g w_j q_{ji}$, where the basis rotation q is needed in order to recover a basis in the admissible set. The same element \hat{g} acts then on the basis $w' = w \cup v$ extending the action on w by sending v_i to gv_i .

The intertwining property of the natural action on $DET_{\lambda\lambda'}$ is exactly what was needed in the definition of the action of $\hat{\mathcal{G}}$ in the Fock bundle over \mathcal{A} . On the other hand, the obstruction to pushing the Fock bundle over $\mathcal{A}/\mathcal{G}_e$ was precisely the class of the extension $\hat{\mathcal{G}} \rightarrow \mathcal{G}$. Thus we have

Theorem 5.2. *The obstruction to pushing forward the trivial bundle $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ to a bundle over the quotient $\mathcal{A}/\mathcal{G}_e$, with the action of $\hat{\mathcal{G}}$ coming from the \mathcal{U}_{res} valued cocycle ω , is the Dixmier-Douady class of the Fock bundle.*

It is clear from the above discussion that we may view the Fock bundle over \mathcal{A} as an associated bundle to the principal bundle $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ defined by the representation of $\hat{\mathcal{U}}_{res}$ in the Fock space of free fermions.

Example Let us take a very concrete example for the discussion above. Let $G = SU(2)$ and the physical space $M = S^1$. Now $\mathcal{A}/\mathcal{G}_e$ is simply equal to G since the gauge class of the connection in one dimension is uniquely given by the holonomy around the circle. Because topologically $SU(2)$ is just the unit sphere S^3 any principal bundle over G is described by its transition function on the equator S^2 . In case of a \mathcal{U}_{res} bundle we thus need a map $\phi : S^2 \rightarrow \mathcal{U}_{res}$ to fix the bundle and the equivalence class of the bundle is determined by the homotopy class of ϕ . The topology of \mathcal{U}_{res} is known: it consists of connected components labelled by the Fredholm index of $P_+ g P_+$, it is simply connected and so the second homotopy is given by $H^2(\mathcal{U}_{res}, \mathbb{Z}) = \mathbb{Z}$. Thus the equivalence class of a principal \mathcal{U}_{res} bundle over $S^3 \equiv \mathcal{A}/\mathcal{G}_e$ is given by the index of the map ϕ .

The principal \mathcal{G}_e bundle $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}_e$ is defined by a transition function $\xi : S^2 \rightarrow \mathcal{G}_e$. This is determined as follows. Since the total space is contractible, we actually have here a universal \mathcal{G}_e bundle over S^3 . Thus the transition function ξ is the generator in $\pi_2(\mathcal{G}_e)$. Such a map can be explicitly constructed. Any point Z on the equator $S^2 \subset S^3$ determines a unique half-circle connecting the antipodes ± 1 . We define $g_Z : S^1 \rightarrow SU(2)$ by first following the great circle through a fixed reference point Z_0 on the equator, as a smooth function of a parameter $0 \leq x \leq \pi - \delta$ (where δ is a small positive constant), from the point $+1$ to the antipode -1 . For parameters $\pi - \delta < x < \pi + \delta$ we let $g_Z(x)$ to be constant, for $\pi + \delta \leq x \leq 2\pi - \delta$ the loop continues from -1 to $+1$ through the point Z on the equator, and finally for $2\pi - \delta \leq x \leq 2\pi$ it is constant. It is easy to see that the set of smooth loops so obtained covers S^3 exactly once and therefore gives a map $g : S^2 \rightarrow \mathcal{G}$ of index one.

Any element of \mathcal{G} is represented as an element of \mathcal{U}_{res} through pointwise multiplication on the fermion field in H . Thus by this embedding we get directly the transition function ϕ for the \mathcal{U}_{res} bundle over $\mathcal{A}/\mathcal{G}_e$.

The index of the map ξ can also be checked using the WZWN action,

$$\text{ind } \xi = \frac{1}{24\pi^2} \int_{S^2 \times S^1} \text{tr} (g^{-1} dg)^3$$

and in the fundamental representation of $G = SU(2)$ this gives $\text{ind } \xi = 1$. For chiral fermions on the circle in the fundamental representation of G this is the same as the index of the map $\phi : S^2 \rightarrow \mathcal{U}_{res}$. This latter index is evaluated by pulling back the curvature form c on \mathcal{U}_{res} to S^2 and then integrating over S^2 . The curvature is defined by the same formula as the canonical central extension of the Lie algebra of \mathcal{U}_{res} . Identifying left-invariant vector fields on the group manifold as elements in the Lie algebra we have

$$c(X, Y) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y].$$

Note that this curvature on \mathcal{U}_{res} is the generator of $H^2(\mathcal{U}_{res}, \mathbb{Z})$.

The Dixmier-Douady class in our example, as a de Rham class in $H^3(\mathcal{A}/\mathcal{G}_e)$, is simply the normalized volume form on S^3 . This is because the third cohomology group of S^3 is one-dimensional and the DD class was constructed starting from the universal bundle $\mathcal{A} \rightarrow S^3$.

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