Spot the Difference:
How to tell when two things are the same (and when they're not!)

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Outline

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Examples of Mathematical Structures

Example	General concept	
$\mathbb{R}^n$	Vector space	$(V, +, \mathbb{R}\text{-mult})$
Symmetries	Group	$(G, \cdot)$
Functions on $\mathbb{R}$	Algebra (non)-example:	$(A, +, \times, \mathbb{R}\text{-mult})$ $(\mathbb{R}^3, \times) \rightsquigarrow$ Lie algebra
$\mathbb{R}$ (cts properties)	Topological space	

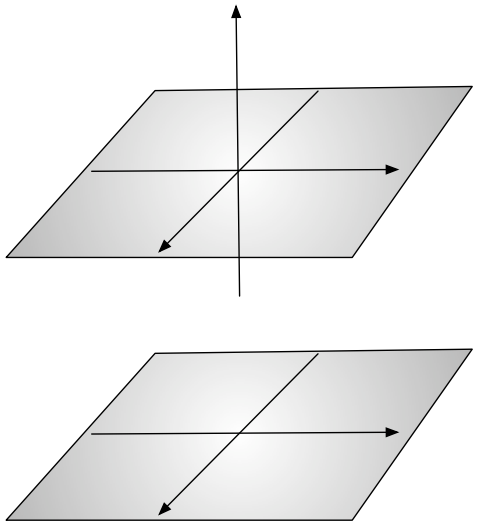
Sameness Problems

Problem

How to come up with a natural notion of "sameness" for these (& other) structures?

- "$=$" is usually too strict

e.g. $\{x\text{-}y \text{ plane}\} \subseteq \mathbb{R}^3$



$\mathbb{R}^2$

$\{x\text{-}y\} \neq \mathbb{R}^2$ but $\{x\text{-}y\} \simeq \mathbb{R}^2$

Similarly: *any* 2D subspace of *any* $\mathbb{R}^n \simeq \mathbb{R}^2$

Sameness Problems

Key Idea

Change object but preserve structure

Want maps $V \to W$ which preserve the vector space (or group, or algebra, or ...) structure

- e.g. NOT $\mathbb{R}^3 \to \mathbb{R}^2$; $(x, y, z) \mapsto (x^2 y, \sin(z)/\sqrt{x})$

Examples:

Object	Maps
Vector spaces	linear maps
Groups	homomorphisms
Algebra	(algebra) homomorphisms
Topological spaces	continuous functions

Sameness Problems

Equivalence: Find maps both ways (i.e. maps with inverses)

Problem

How can you tell if 2 objects are equivalent or not?

- $A \simeq B$: Find one
- $A \not\simeq B$: Extract info which doesn't change under equivalence and show it's different for A and B

$\rightsquigarrow$ invariants

Example: (finite-dimensional) vector spaces:

$$V \simeq W \Leftrightarrow \dim V = \dim W$$

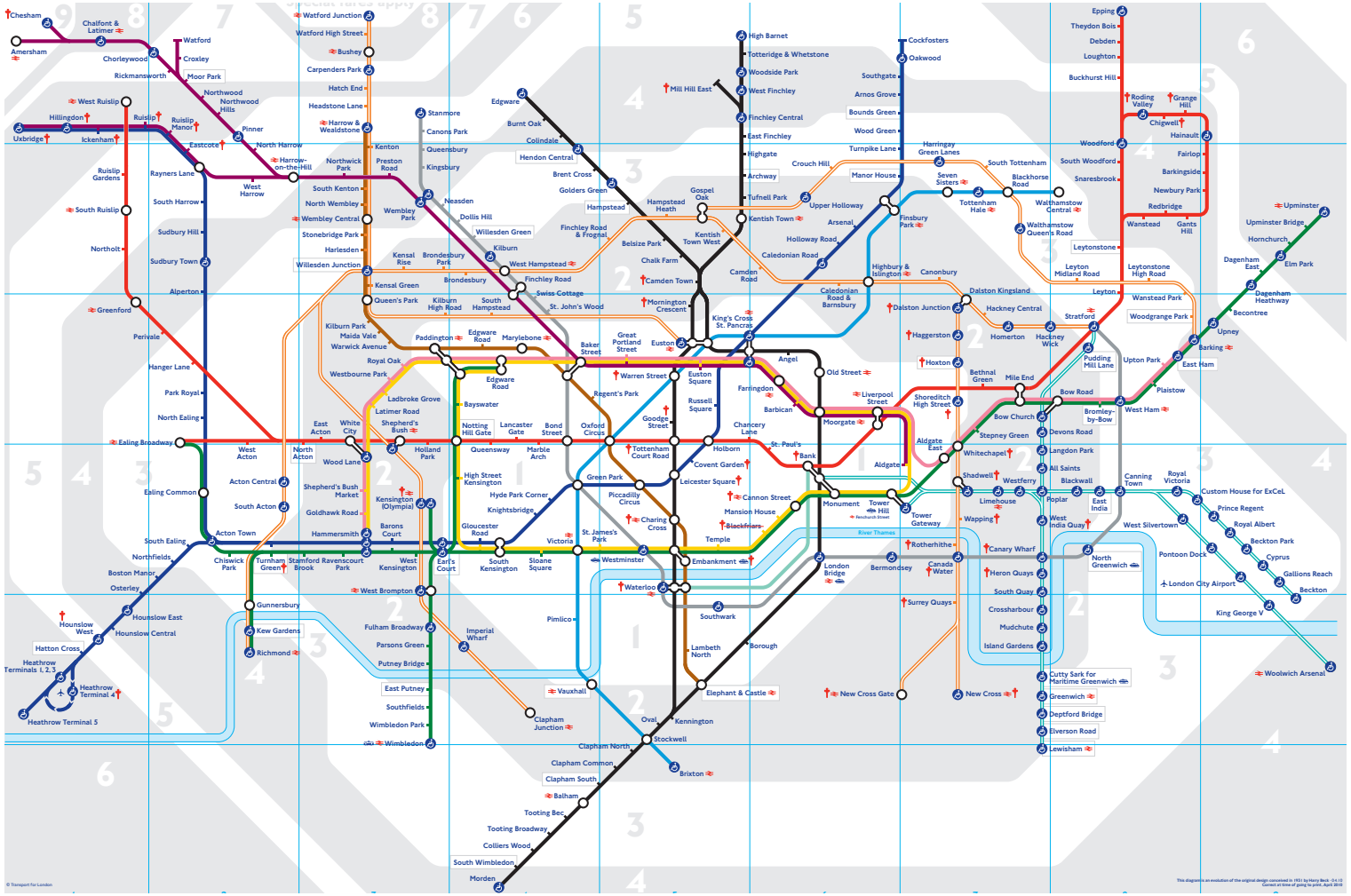
Outline

Topology

Idea: Throw away notions of measurement, distances, etc. . .

Keep ideas of “closeness” and “connectedness”

Example:



Topology

Equivalency condition: Continuous deformations

(*homeomorphisms*)

- i.e. $X \xrightarrow{f} Y$ and $Y \xrightarrow{f^{-1}} X$ continuous

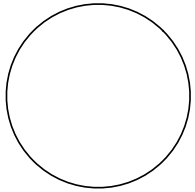
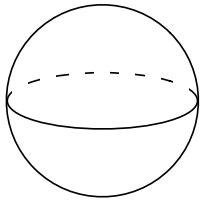
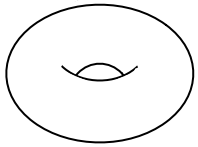
Example: $\mathbb{R} \to \mathbb{R};\ x \mapsto x^3$ vs $\mathbb{R} \to \mathbb{R};\ x \mapsto \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Manifolds

Manifolds are important examples of topological spaces

Roughly: A manifold is a topological space which locally looks like (i.e. is homeomorphic to) $\mathbb{R}^n$

Examples:

1D	S^1	
2D	S^2	
	$\mathbb{T}^2$	
	Other surfaces	
3D	$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$	
4D	The Universe!	
10D	The Universe?	

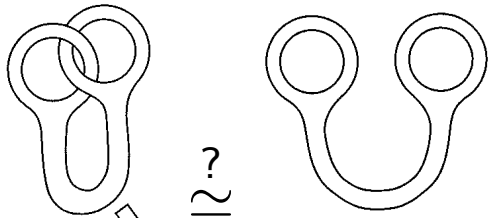
Algebraic Topology

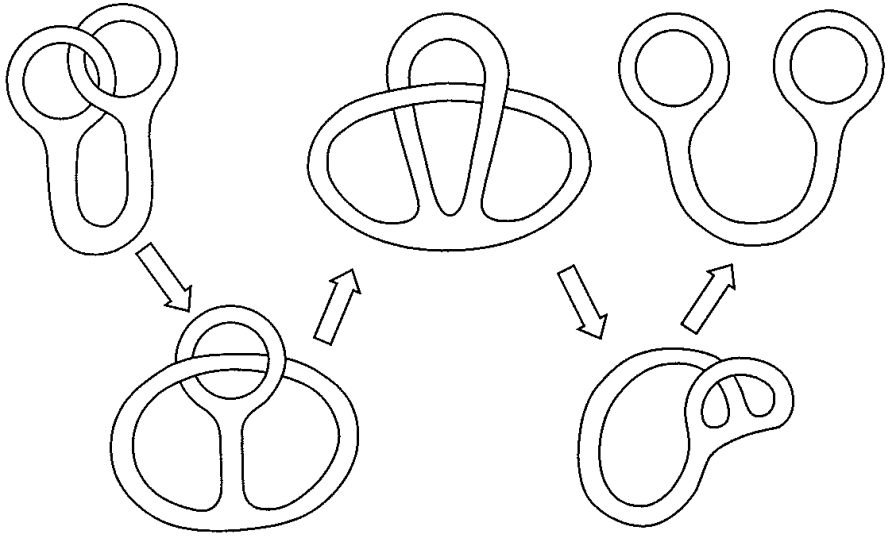
Question

How can you tell if 2 manifolds are homeomorphic?

Answer: Look for invariants!

Example Question: $S^2 \overset{?}{\simeq} \mathbb{T}^2$

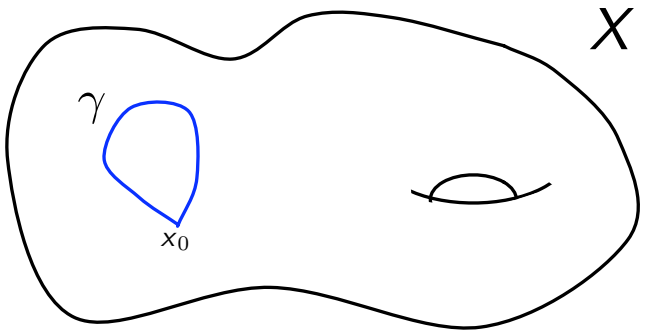
Note: It's not necessarily obvious. Consider 



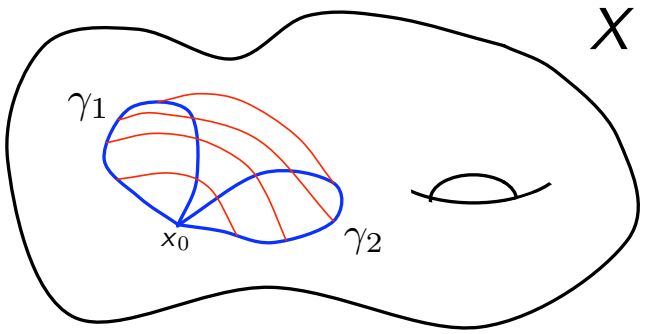
The Fundamental Group

Consider loops:

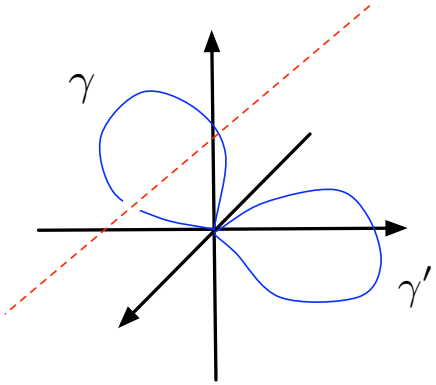
$\{\gamma\colon [0,1] \to X \mid \gamma(0) = \gamma(1) = x_0\}$



$\gamma_1 \sim \gamma_2 \iff \gamma_1 \xrightarrow[\text{deform}^n]{\text{cts}} \gamma_2$



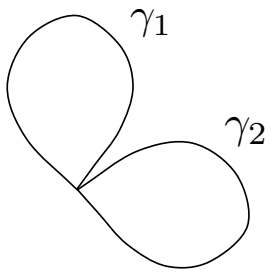
e.g. $\gamma \not\sim \gamma'$



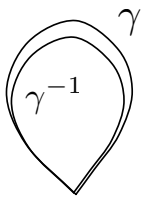
The Fundamental Group

Define:

- $\gamma_1 * \gamma_2 = \gamma_1$ then γ_2



- $\gamma^{-1} = \gamma$ backwards



$\Longrightarrow$ It's a group!

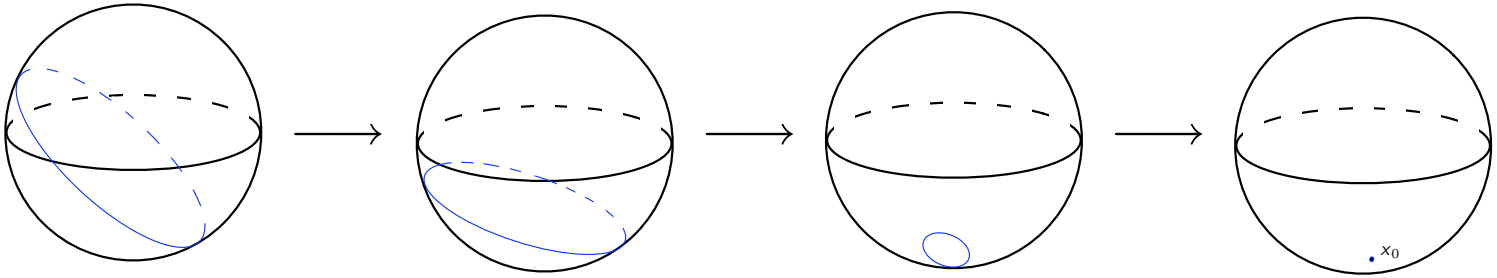
The Fundamental Group, $\pi_1(X)$

Important Property: $X \simeq Y \Longrightarrow \pi_1(X) \simeq \pi_1(Y)$

The Fundamental Group

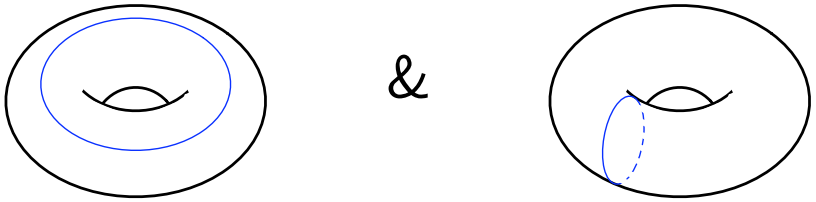
Back to question: $S^2 \overset{?}{\simeq} \mathbb{T}^2 \quad \rightsquigarrow$ is $\pi_1(S^2) \overset{?}{\simeq} \pi_1(\mathbb{T}^2)$

$\pi_1(S^2)$:



$\Longrightarrow \pi_1(S^2) \simeq \{0\}$ (i.e. S^2 is simply connected)

$\pi_1(\mathbb{T}^2)$:



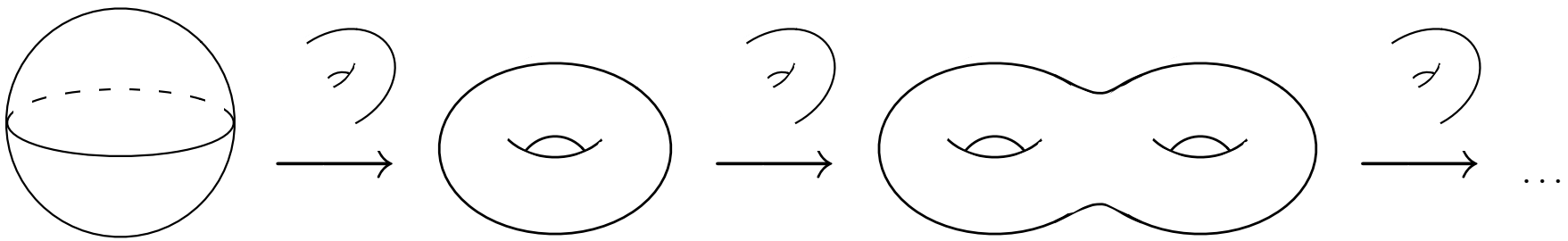
$\Longrightarrow \pi_1(\mathbb{T}^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$

$$\pi_1(S^2) \not\simeq \pi_1(\mathbb{T}^2) \Longrightarrow S^2 \not\simeq \mathbb{T}^2$$

Classification of Surfaces

In general π_1 doesn't completely characterise X

- 2D: it does

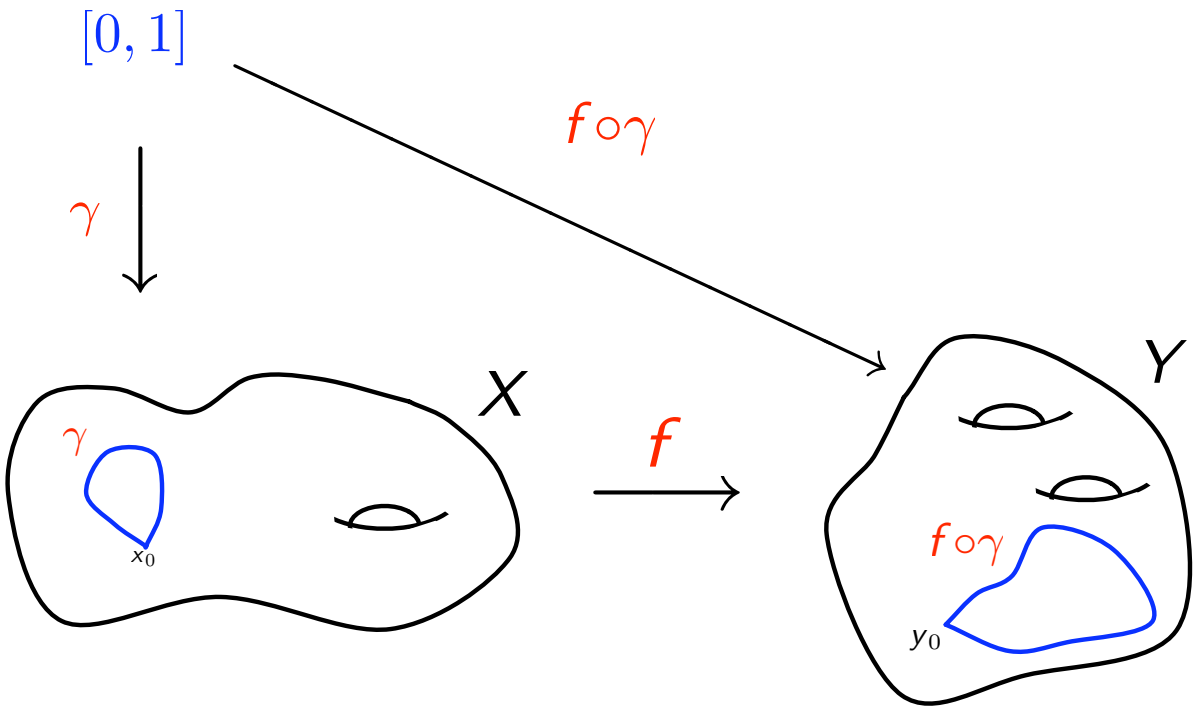


Interesting Question: What about higher dimensions?

$$\text{e.g. } \pi_1(X^3) = 0 \overset{?}{\Longrightarrow} X^3 \simeq S^3$$

More about π_1

Important Property: $f\colon X \to Y$ induces $f_*\colon \pi_1(X) \to \pi_1(Y)$



- π_1 is an example of a functor

Outline

Category Theory

- Formalism for talking about structures and maps which preserve them
- A category consists of
  - Objects
  - Arrows between them

Examples:

Top	Topological spaces and continuous maps
Grp	Groups and group homomorphisms
Vect	Vector spaces and linear maps
Cat	Category of Categories!

- Invariants for topological spaces $\longleftrightarrow$ Functors **Top** $\rightarrow$ **Grp**

More functors: π_n, H_n