Predicting turbulence

T. W. Mattner

12 August 2009

Turbulence

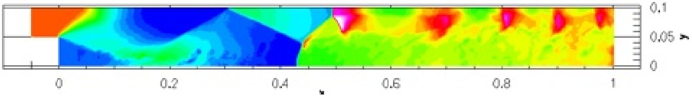
- Turbulence is a state of flow characterised by unsteady, three-dimensional fluid motion over a wide range of spatial and temporal scales.

Turbulence

- Turbulence enhances mixing and increases energy dissipation.

Turbulence

- Turbulence enhances mixing and increases energy dissipation.



Dimotakis *et al.*, Caltech

Turbulence

- Turbulence enhances mixing and increases energy dissipation.

NASA

Navier–Stokes equations

- Mass conservation:

$$\frac{\partial u_j}{\partial x_j} = 0$$

Navier–Stokes equations

- Mass conservation:

$$\frac{\partial u_j}{\partial x_j} = 0$$

- Newton's Law:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2}$$

Navier–Stokes equations

- Mass conservation:

$$\frac{\partial u_j}{\partial x_j} = 0$$

- Newton's Law:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2}$$

- The Reynolds number is

$$Re = \frac{UL}{\nu}.$$

Solution

Solution

- Clay Mathematics Institute Millenium Problem:

Since we don't even know whether these solutions exist, our understanding is at a very primitive level.

Fefferman, 2000

Solution

- Clay Mathematics Institute Millenium Problem:

  Since we don't even know whether these solutions exist, our understanding is at a very primitive level.

  Fefferman, 2000

- See http://www.claymath.org/millennium/

Direct numerical simulation (DNS)

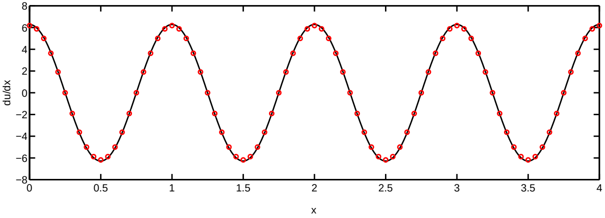
- Use numerical methods to obtain an approximate solution.
  - finite difference
  - finite element
  - finite volume
  - spectral
  - meshless
- Must resolve large and small (fine) scales simultaneously.

Resolution of finite difference schemes

- Let $x_i = i\,\Delta x$ and $u_i = u(x_i)$, where i is an integer.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

- Suppose $u(x) = \sin(2\pi x)$.

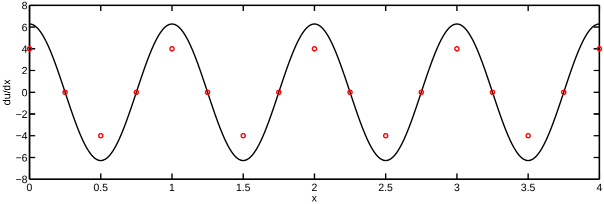


Resolution of finite difference schemes

- Let $x_i = i\,\Delta x$ and $u_i = u(x_i)$, where i is an integer.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

- Suppose $u(x) = \sin(2\pi x)$.

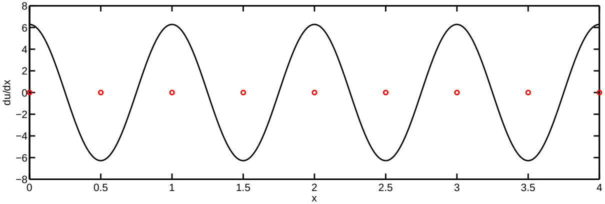


Resolution of finite difference schemes

- Let $x_i = i\,\Delta x$ and $u_i = u(x_i)$, where i is an integer.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

- Suppose $u(x) = \sin(2\pi x)$.



Scale separation in turbulence

- Let ϵ be the energy dissipation rate per unit mass. It is observed that the finest length scale η depends on ϵ and the viscosity ν. Dimensional analysis implies

$$\eta \sim \left(\frac{\nu^3}{\epsilon} \right)^{1/4}$$

- It is observed that the dissipation is determined by U and L, which characterise the large-scale flow. Dimensional analysis implies

$$\epsilon \sim \frac{U^3}{L}$$

- Thus

$$\frac{\eta}{L} \sim Re^{-3/4}$$

Computational work for DNS

- The number of grid points needed is

$$N \sim \left(\frac{L}{\Delta x}\right)^3 = Re^{9/4}$$

- The number of time steps needed is

$$M \sim \frac{T}{\Delta t} \sim \frac{UT}{\Delta x} \sim Re^{3/4}$$

- The computational work is proportional to

$$MN \sim Re^3$$

Computational time for DNS

- For air, $\nu \approx 8 \times 10^{-6}$ m^2/s.

	U	L	Re	Wall-clock time
Reference			1×10^4	1 day

Computational time for DNS

- For air, $\nu \approx 8 \times 10^{-6}$ m^2/s.

	U	L	Re	Wall-clock time
Reference			1×10^4	1 day
B747 wing	250 m/s	0.1 m	3×10^6	$74,000$ years

Computational time for DNS

- For air, $\nu \approx 8 \times 10^{-6}$ m^2/s.

	U	L	Re	Wall-clock time
Reference			1×10^4	1 day
B747 wing	250 m/s	0.1 m	3×10^6	$74,000$ years
Hurricane	200 km/h	1 km	7×10^9	9×10^5 billion years

Computational time for DNS

- For air, $\nu \approx 8 \times 10^{-6}$ m^2/s.

	U	L	Re	Wall-clock time
Reference			1×10^4	1 day
B747 wing	250 m/s	0.1 m	3×10^6	$74,000$ years
Hurricane	200 km/h	1 km	7×10^9	9×10^5 billion years

- The Universe is about 14 billion years old.
- DNS is not feasible for such flows.

Large-eddy simulation (LES)

- Smooth flow variables using a filter

$$\overline{\phi}(\boldsymbol{x}) = \int \phi(\boldsymbol{x}')\, G(\boldsymbol{x} - \boldsymbol{x}'; \Delta)\, \mathrm{d}\boldsymbol{x}'$$

- Apply to Navier–Stokes equations

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i \overline{u}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \overline{u}_i}{\partial x_j^2} - \frac{\partial T_{ij}}{\partial x_j}$$

- The subgrid-scale (SGS) stress is

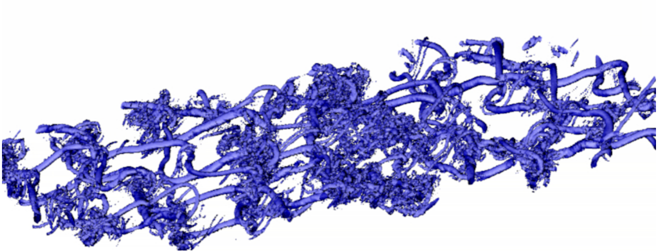
$$T_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$$

Subgrid-scale motion

- The vorticity is

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

It is related to the rotation of fluid elements.



An exact solution

- Vorticity is transported by the fluid. It is intensified by stretching and diffused by viscosity.
- An exact solution of the Navier–Stokes equation in cylindrical coordinates (r, θ, z) is

$$u_r = -\frac{1}{2}ar, \quad u_z = az, \quad u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-ar^2/4\nu}\right)$$

  This is an axisymmetric solution.
- The vorticity is aligned in the z direction

$$\omega_z = \frac{a}{2\nu}e^{-ar^2/4\nu}$$

An asymptotic solution

- Introduce stretched coordinates

$$\rho = e^{at/2} r \quad \text{and} \quad \tau = \frac{e^{at} - 1}{a}$$

- The vorticity is

$$\omega_z = \sum_{n=-\infty}^{\infty} \omega_n(\rho, \tau) e^{in\theta},$$
$$\omega_n(\rho, \tau) = f_n(\rho)\, e^{-in\Omega(\rho)\tau - \nu^2 \Omega'^2(\rho)\tau^3/3}$$

- The error is $O(\tau^{-1})$

Stretched-vortex subgrid-scale stress model

- Model turbulence as a homogeneous ensemble of nonaxisymmetric stretched vortices.
- Start with the vorticity correlation function

$$
\begin{aligned}
W_{ij}(\boldsymbol{\rho}) &= \overline{\omega_i(\boldsymbol{x})\omega_j(\boldsymbol{x}+\boldsymbol{\rho})} \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi} F_{ij}\, P(\alpha,\beta,\gamma)\, \sin\alpha\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma
\end{aligned}
$$

$$
\begin{aligned}
F_{ij} = \frac{\sum_m l_m}{L^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & \omega_z^{(m)}(x_1', x_2', t)\,\omega_z^{(m)}(x_1'+\rho_1', x_2'+\rho_2', t) \\
& \times E_{3i}E_{3j}\, \mathrm{d}x_1'\, \mathrm{d}x_2'
\end{aligned}
$$

Stretched-vortex subgrid-scale stress model

- Assume that vortices are aligned with the unit-vector $\boldsymbol{e}^v$. Then it is found that

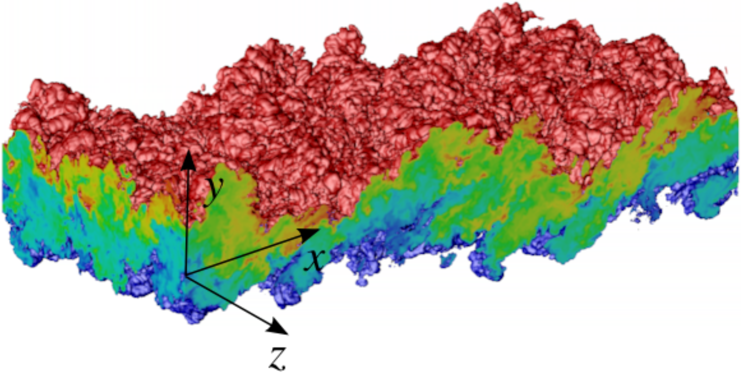
$$T_{ij} = \overline{u_i u_j} = K(\delta_{ij} - e_i^v e_j^v)$$
$$K = \int_{k_c}^{\infty} E(k)\,\mathrm{d}k$$

- For nonaxisymmetric stretched-spiral vortices, it is found that

$$E(k) = \mathcal{K}_0 \epsilon^{2/3} k^{-5/3} e^{-2k^2\nu/3a}$$

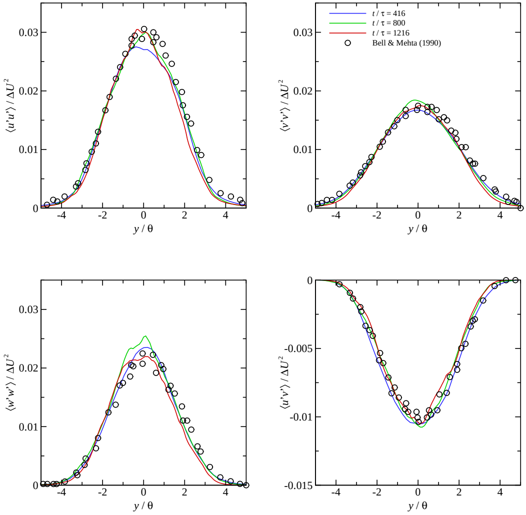
- The parameters $\mathcal{K}_0\epsilon^{2/3}$, a and $\boldsymbol{e}^v$ are estimated from the resolved-scale flow.

Turbulent mixing layers

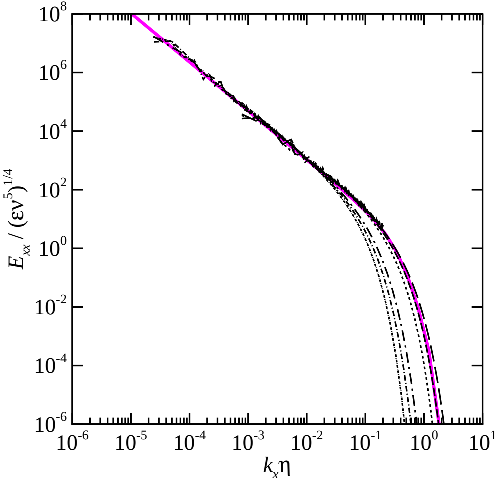


Turbulent mixing layers

Reynolds-stress tensor – *cf.* Bell & Mehta (1990)



Velocity spectra



Conclusion

- Turbulence is governed by the Navier–Stokes equations
- It is presently impossible to obtain solutions to high Reynolds number turbulent flows.
- Nevertheless, large-eddy simulations using subgrid-scale models produce statistics that are consistent with experimental data.
- Further work:
    - Variable density flows
    - Reacting flows
    - Multiphase flows
    - Transition

Conclusion

I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.

Horace Lamb, 1932