Is the price *really* right?

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Let X be a the amount of money you might win in a game of chance.

- Q. What is the value of X today?
- A. (First year) It is the amount you would be willing to pay today to win X at a later time.
- A. (PhD Student) $E[X] := \int_\Omega X(\omega) d\mathbb{P}(\omega)$, the integral of X with respect to the probability measure, (of course).

Is there a difference in these answers?

Outline

Risk and Pricing

- Today I will talk about pricing, measures of risk and nonlinear expectations.
- Essentially, a key question is:
  Given a future ***random*** *payoff X, what are you willing to pay* ***today*** *for X?*
- The converse question is “How *risky* is X?”.
- We begin with a quick summary of some ways people have answered this question in the past.

- Historically, people first assumed the *mathematical expectation* $E[X]$ was the ‘correct’ price for X.
- This is the value that, if the game were played repeatedly, in the long run you would neither win nor lose by playing.
- Small Problem: Which probabilities should we use? (Frequentist/Bayesian???)
- Unfortunately, this doesn’t seem to work...

Problem 1: St. Petersburg Paradox

- Suppose $E(X)$ is what you are willing to pay today to receive the random amount X.
- Consider the following game:
  - Take a fair coin, flip it until a tail appears. Let n be the number of heads observed.
  - The random amount you will receive is $X = 2^n$.
  - Hence $E[X] = \sum_n 2^n \mathbb{P}(n) = \sum_n 2^n (1/2)^n = \sum_n 1 = \infty$.
- Problem: People are not willing to pay ∞ (or any other large amount) to play this game.

Solution (Daniel Bernoulli):

- People don't care about the amount of money X, they care about the utility of this money $u(X)$.
- u should be concave and increasing

This was then extended by von Neumann, et alii.

We can define the *certainty equivalent* of X: the fixed amount you are willing to trade X for.

$$CE(X) := u^{-1}E[u(X)]$$

Problem 2: Ellsberg Paradox

- We have two containers, each containing 100 pieces of paper (red & black).
- Container 1 has 49 red, 51 black.
- Container 2 is unknown.
- A player can choose a container, draw one paper:
  - if red they win, else nothing.

Problem 2: Ellsberg Paradox

Empirically:

- Most people choose Container 1.
- Most people would still choose Container 1 if we said they win for black, nothing for red.
- This is inconsistent with *any* probability of choosing a red piece of paper from Container 2.

Hence $CE(X)$ does not seem to describe behaviour.

Problem 3: Framing

Consider the following two games:

- You give me \$5.
  We flip a coin – Heads you get \$10, Tails nothing.
  Call this amount X_1.
- You give me \$0.
  We flip a coin – Heads you get \$5, Tails you give me \$5.
  Call this amount X_2.

Clearly these games are the same – so if you like one you should like the other. That is,

$$CE(X_1) \geq 5 \Rightarrow CE(X_2) = CE(X_1 - 5) \geq 0.$$

Problem 3: Translations

For pricing, we want the price to move correctly under translation (ignoring interest rates).

- The price of $(X + \$10)$ should be (the price of X)+\$10.
- If $CE(X) = u^{-1}E[u(X)]$ should give the price,

$$u^{-1}E[u(X + c)] = u^{-1}E[u(X)] + c$$

  for all $c \in \mathbb{R}$.
- From the Kolmogorov-de Finetti theorem on associative means, this only works if u is linear or exponential.

For these reasons, people have tried various other solutions:

- Markowitz (1952) assumes people only care about the mean and variance of X
- A problem here is that we can construct $X_1 \geq X_2$ almost surely, but X_2 be preferred to X_1.

What other approaches are there?

Pricing and Hedging

A fundamental approach to pricing in most of mathematical finance comes through the idea of hedging:

- Consider a random outcome X,
- Suppose there is a portfolio of assets *with given prices* that gives the same payoffs as $-X$
- Combining this portfolio with X gives a payoff of zero, so should cost nothing
- Then the price of X should be the negative of the price of the portfolio.

- If we know the prices of enough assets to do this for *any* payoff X, then the market is known as *complete*.
- Assuming prices in the market are linear, this can be used to create a (unique) probability distribution π such that the price of X is $E_\pi[X]$, (ignoring interest rates).
- However, this distribution may not be the ‘real world’ probability distribution.
- Most of mathematical finance (eg. Black-Scholes option pricing) begins with this approach.

Pricing or Risk management?

- This theory is good for pricing, but it doesn't work so well for risk assessment.
- Here, we want to know if a risky position is *acceptable*.
- Alternatively, we want to know how much we need to keep in reserves in case of a negative outcome.
- For this problem, we still want translation invariance, but we don't usually want linearity – we want our definition of risk to encourage diversification.

Measures of Risk

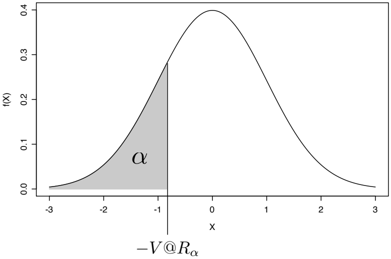
- We have been discussing how to measure possible winnings (X).
- We now focus on possible losses ($-X$).
- This is because 'high risk' loosely corresponds to large losses/low winnings.
- Rather than be precise about what we mean by 'risk' we will rather talk about 'measures of risk'.

Value at Risk

We define the Value at Risk at level α ($V@R_\alpha$) by

$$V@R_\alpha(X) = \inf\{x : \mathbb{P}(X < -x) \leq \alpha)\}$$

(note the $-x$)



$V@R$ has become a standard risk measure, (eg. Basel II, Petroleum industry)

Example: A Dodgy Bank

- Suppose a bank owns a risky asset, with payoff from a $N(1, 2)$ distribution.
- The bank has a subsidiary it can share assets with.
- The owner of risky assets must store capital equal to the $V@R_{0.1}$ of their position.

How should the bank proceed to minimise its total capital requirement?

Option 1: Hold on to the asset

Then the payoff for the bank is from a $N(1,2)$ distribution, and so the V@R is

$$V@R_{0.1} = -\Phi^{-1}(0.1; 1, 2) = 0.8123.$$

Option 2: Do some clever trades

Make the following agreements with the subsidiary.

- If the asset pays less than -0.9, the main bank takes it
- If the asset pays between -0.9 and 0, the subsidiary takes it
- If the asset pays above 0, the main bank takes it.

The V@R for the main bank is 0, for the subsidiary is 0.2439.

- Using Strategy 2, we have a total capital requirement of 0.2439, as opposed to the ‘natural’ requirement of 0.8123. Nothing illegal has happened here!
- Therefore, by rearranging the books appropriately, the capital requirement has been decreased.
- More elaborate schemes (with correlated assets etc...) allow it to be decreased further.

- Provided all random outcomes are normally distributed, V@R is convex (so this won't work).
- When outcomes are not normally distributed, $V@R$ is not convex.
- Most financial problems involve non-normal outcomes (from options etc...)
- $V@R$ allows the risk to be disaggregated in such a way so as to lower the apparent risk!
- This is counterintuitive and open to exploitation.

Coherent Measures of Risk

Artzner, Delbaen, Eber & Heath (1999) define "Coherent measures of risk":

$$\rho : L^1(\mathcal{F}) \to \mathbb{R}$$

where

- (Monotonicity) $X^1 \geq X^2$ with probability 1 implies

$$\rho(X^1) \leq \rho(X^2)$$

- (Translations) $\rho(X + c) = \rho(X) - c$ for all $c \in \mathbb{R}$.
- (Positive Homogeneity) For all $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$
- (Subadditivity) $\rho(X^1 + X^2) \leq \rho(X^1) + \rho(X^2)$

Convex Measures of Risk

Frittelli & Rosazza Gianin (2002) and Föllmer & Schied (2002) (independently) defined "Convex Measures of Risk"

$$\rho : L^1(\mathcal{F}) \to \mathbb{R}$$

where

- (Monotonicity) $X^1 \geq X^2$ with probability 1 implies

$$\rho(X^1) \leq \rho(X^2)$$

- (Translations) $\rho(X + c) = \rho(X) - c$ for all $c \in \mathbb{R}$.
- (Convexity) For all $\lambda \in [0, 1]$,

$$\rho(\lambda X^1 + (1 - \lambda)X^2) \leq \lambda\rho(X^1) + (1 - \lambda)\rho(X^2)$$

In many cases, we may want

- (Constants) For all $c \in \mathbb{R}$,

$$\rho(c) = -c.$$

Given these assumptions, we can then interpret $\rho(X)$ as

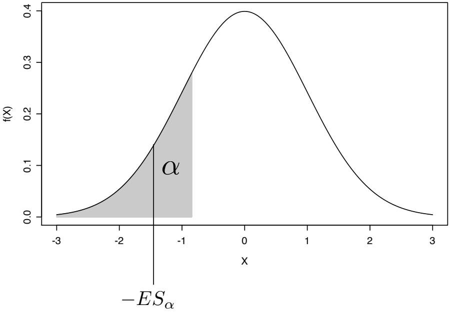
> *the smallest amount of money that I need to add to a risky position to make it acceptable*

where 'acceptable' means $\rho(X) \leq 0$.

Example: Expected shortfall

We define the expected shortfall at level α (ES_α) by

$$ES_\alpha(X) = E[-X | X \leq -V@R_\alpha(X)]$$



Example: Expected shortfall

- ES is a coherent (and convex) risk measure.
- Despite its advantages, ES is less common than $V@R$.
- Various computational tools have been developed to estimate both ES and $V@R$.

How to create risk measures?

From a mathematical perspective, we would like a representation of risk measures in terms of simpler objects.

As is common in convex analysis, we *can* find a nice representation here:

$$\rho(X) = \sup_{\mathbb{Q}\sim\mathbb{P}}\{E_{\mathbb{Q}}[-X] - \beta(\mathbb{Q})\}$$

where β is some ‘penalty’ function and the supremum is taken over some set of probability measures absolutely continuous with respect to $\mathbb{P}$.

Example: Entropic Risk

We can define the entropic risk in either of two ways:

$$\rho(X) = \gamma \ln E\left[e^{-X/\gamma}\right]$$

or

$$\rho(X) = \sup_{\mathbb{Q}\sim\mathbb{P}}\{E_{\mathbb{Q}}[-X] - h(\mathbb{Q})\}$$

where

$$h(\mathbb{Q}) = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = E_{\mathbb{Q}}\left[\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

is the ‘relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$’.

Example: Entropic Risk

The entropic risk has various nice properties

- it is just the negative of the certainty equivalents defined above,
- it has a simple generalisation to multiple time periods:

$$\rho_t(X) = \gamma \ln E \left[e^{-X/\gamma} \middle| \mathcal{F}_t \right],$$

However, it is also very difficult to estimate without making a lot of assumptions. (It is extremely sensitive to negative outliers in data.)

To create coherent risk measures, an alternative approach is to use the ‘Choquet integral’.

Definition

Let ν be a (nice) monotone set function $[0, \infty) \supset \mathcal{F} \to \mathbb{R}$, then for any (nice) nonnegative function f, we can write

$$(C) \int_{[0,\infty)} f \, d\nu = \int_{[0,\infty)} \nu\{s : f(s) \geq x\} \, dx$$

This can also be extended to allow possibly negative-valued functions. If ν is a probability measure, this corresponds with the usual expectation.

If we have a set function ν which is '2-modulating', that is

$$\nu(A \cup B) \leq \nu(A) + \nu(B) - \nu(A \cap B)$$

and if $\nu([0, \infty)) = 1$, then

$$\rho(X) = (C) \int (-X)\, d\nu$$

is a coherent risk measure.

Some applications

Risk measures have been used to study various problems, including

- pricing in incomplete markets,
- financial regulation,
- game theory,
- stochastic optimal control,
- risk-sharing,
- and more to come!

Unanswered Questions

A variety of problems still remain

- How do we generalise these risk measures to multiple time periods in a consistent way (so that we don't keep changing our mind)?
- What types of risk measures can be estimated efficiently/robustly in a model-free way?
- What is the best risk measure to use in practice for problem ....?