Infinite numbers: what are they and what are they good for?

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2010

A funny way of writing numbers

Take a natural number $n \geq 2$ as a base.

Given any natural number, write it as a sum of powers of n, do the same for all the exponents in that expression, and so on.

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Examples with base 2:

$33 = 2^5 + 1 = 2^{2^2+1} + 1$

$266 = 2^8 + 2^3 + 2 = 2^{2^3} + 2^{2+1} + 2 = 2^{2^{2+1}} + 2^{2+1} + 2$

Goodstein sequences

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The first term k_2 is k itself.

For $n \geq 2$, given the number k_n, if $k_n = 0$, then stop. Otherwise,

- write k_n using base n,
- replace every n in that expression by $n + 1$,
- and subtract 1.

This produces k_{n+1}.

Example: $k = 3$

$$3_2 = \mathbf{3} = 2 + 1$$

$$3_3 = 3 + 1 - 1 = \mathbf{3}$$

$$3_4 = 4 - 1 = \mathbf{3}$$

$$3_5 = 3 - 1 = \mathbf{2}$$

$$3_6 = 2 - 1 = \mathbf{1}$$

$$3_7 = 1 - 1 = \mathbf{0}$$

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This number is of the order of $10^{121,210,700}$.

The age of the universe is about $4 \cdot 10^{17}$ seconds.
The number of atoms in the universe is about 10^{80}.

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Theorem (Goodstein 1944). For every natural number k, the Goodstein sequence $k_2, k_3, k_4, \ldots$ eventually reaches zero.

Goodstein proved his theorem using infinite numbers called ordinals.

Reuben Louis Goodstein (1912–1985)

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$1 = \{0\} = \{\varnothing\}$
$2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$
$3 = \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
$4 = \{0, 1, 2, 3\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$
$\vdots$

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$$\{0, 1, 2, 3, \ldots, \omega\} = \omega \cup \{\omega\}.$$

In general, if α is a number, then the next number is $\alpha \cup \{\alpha\}$.

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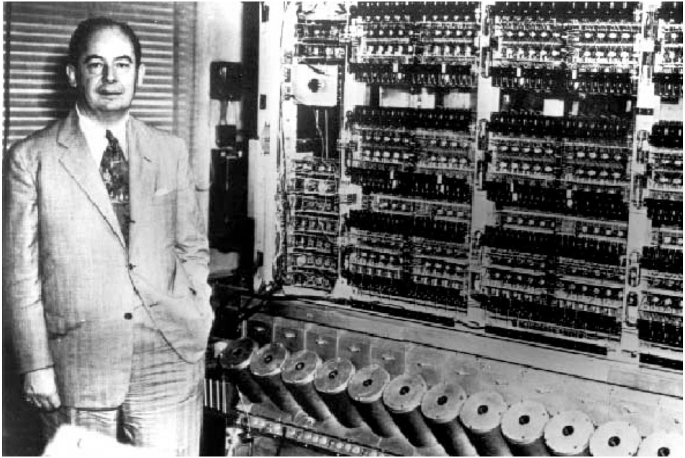
In the 1920s, John von Neumann identified the crucial properties of these sets and turned them into the definition of an ordinal.

- They are **well-ordered**: every nonempty subset has a smallest element.
- Every element *equals* the set of elements smaller than it.



John von Neumann (1903–1957)
The first computer of the Institute for Advanced Study, Princeton

Well-ordered sets

Recall: A **partially ordered set** is a set A with a relation (a subset of $A \times A$) denoted $\leq$, such that:

- $a \leq a$ for all $a \in A$ (reflexivity).
- If $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetry).
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Thus, α is ordered by $\in$ or, equivalently, $\subset$.

Ordinal arithmetic

We can add and multiply ordinals:

$\alpha + \beta$	α followed by β
$\alpha \cdot \beta$	replace each element of β by α

Clearly, $\alpha + 0 = 0 + \alpha = \alpha$ and $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

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Any two ordinals are comparable.

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Theorem. Every well-ordered set is isomorphic to a unique ordinal.
So there are "enough" ordinals.

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By well-ordering, the sequence of ordinals must terminate, so the Goodstein sequence must terminate, q.e.d.

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The usual foundation for the theory of the natural numbers is:

Peano's axioms (1890s). **1.** 0 is a natural number.
2. Every natural number n has a successor n^+, which is also a natural number.
3. 0 is not the successor of any natural number.
4. Distinct numbers have distinct successors.
5. If $P(0)$ is true, and whenever $P(k)$ is true, $P(k^+)$ is also true, then $P(n)$ is true for all natural numbers n.

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There is even a serious claim that Wiles's proof of Fermat's Last Theorem can be carried out in Peano arithmetic.

Kirby and Paris

Theorem (Kirby and Paris 1982). Goodstein's theorem *cannot* be proved (or disproved) in Peano arithmetic.

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We conclude that Peano's axioms do not capture all truths about natural numbers—if we regard the axioms of set theory as "true".

Some true statements about the natural numbers cannot be proved without the use of infinite numbers.