How round is your triangle, square, pentagon,...?

Barry Cox

The University of Adelaide

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Abstract

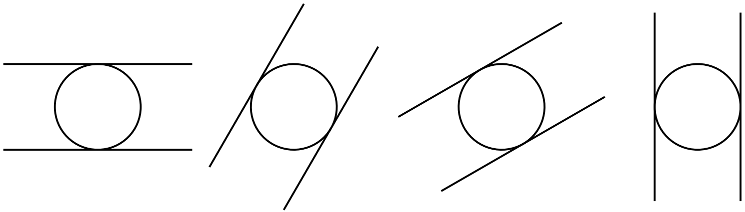
Most of us are familiar with the problem of making circular holes in wood or other material. For smaller diameter holes we typically use a drill, and for larger diameter holes a spade-bit, hole-saw or plunge router may be used. However for some applications, like mortise-and-tenon joints, what is needed is a tool that will produce a hole with a cross-section that is something other than a circle. In this talk we look at curves that may be used as the basis for a device that will produce holes with a cross-section of an equilateral triangle, square, or any regular polygon. Along the way we will touch on areas of Engineering, Algebra, Geometry, Calculus, Gothic Art and Architecture.

Question

How can we test if a given convex curve is circular?

It is necessary that a circle have a constant width.

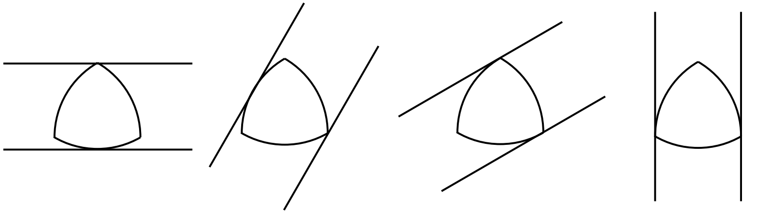
However, is checking the width of the curve from any orientation a sufficient test for roundness?



Curves of constant width

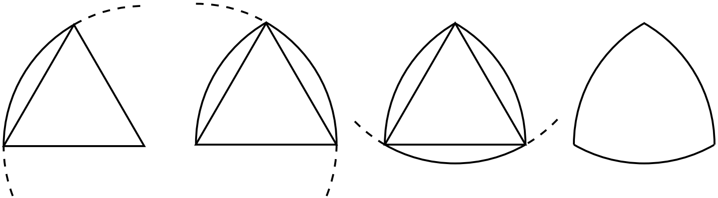
It turns out that the answer is no. There are a whole family of curves that have the constant width property without being round.

The best-known and simplest of these is the Reuleaux triangle, which may be constructed from an equilateral triangle and a pair of compasses. However it is easy to construct such curves from regular pentagons, heptagons or any polygon with an odd number of sides.



Constructing a Reuleaux triangle

1. Start with an equilateral triangle with the edge length of the required width
2. Set the compass width to the triangle edge length
3. Centering the compass on each vertex in turn, connect the other two vertices with a circular arc.



Solids of constant width

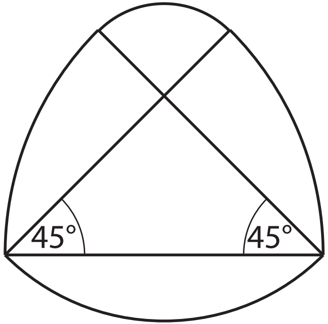
Applications - coins

Applications - square drill

Family of curves of constant width

There are many families of such curves.

One family can be generated from the equilateral triangle if two of the vertices are reduced to some angle less than 60° without shortening the sides.



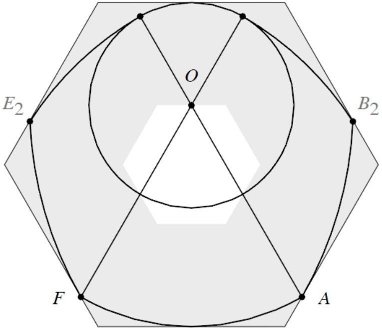
Drilling a perfect square

Other polygons with an even number of sides

The same can be applied to

- hexagons (right)
- octagons
- decagons

and the conjecture is that any even sided polygon can be handled with a similar sort of rotor.



NB: Not a curve of constant width!

Odd sided polygons

After looking at regular n-gons for n even, the question naturally arises, what about when n is odd?

In this case, there are no parallel sides and thus the idea that works for the square and was successfully extended for the hexagon, etc, will not work here.

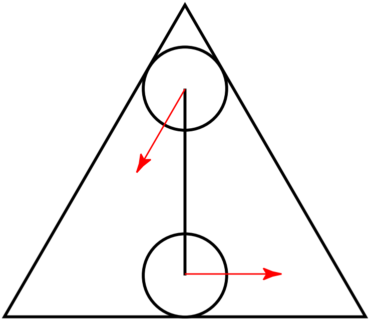
However rotors do exist and in fact the simplest of these was known to Reuleaux!

Drilling a triangle - the idea

Start with two congruent circles of arbitrary radius.

Move them around the trammel maintaining a fixed distance between the centres.

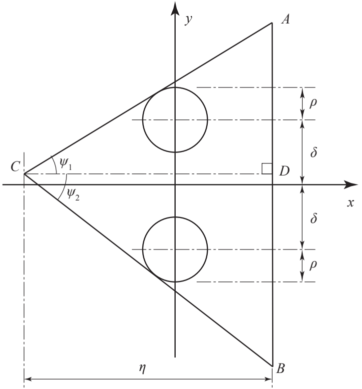
What is the shape of the rotor that provides stable contact with the trammel sides at all times?



Drilling a triangle

Thinking about this mathematically (i.e. in reverse)

- Start with two congruent circles
- Consider a triangle ABC
- Fix AC tangential to the upper circle
- Fix BC tangential to the lower circle
- Now rotate $\triangle ABC$



What is the shape generated by the free side AB?

Calculating envelopes

Curves generated from families of other curves are called envelopes.

One way to calculate envelopes is by considering a parametric representation of the generating family of curves. Assuming you have a one-parameter family then your representation will contain two parameters

1. the standard parametric variable for the curve
2. a parameter that generates the family

If we denote these parameters with the variable p and q then in 2D we have a family of curves given by

$$(x(p, q), y(p, q))$$

Calculating envelopes

However our envelope is a simple curve and thus will be a function of one parameter. Hence we need an extra constraint to reduce the degrees of freedom of the above representation by one.

This is provided by the following condition

$$\frac{\partial x(p,q)}{\partial p}\frac{\partial y(p,q)}{\partial q} = \frac{\partial x(p,q)}{\partial q}\frac{\partial y(p,q)}{\partial p}$$

Generally this allows us to find one parameter as a function of the other, say $q = Q(p)$, and hence the envelope is given by

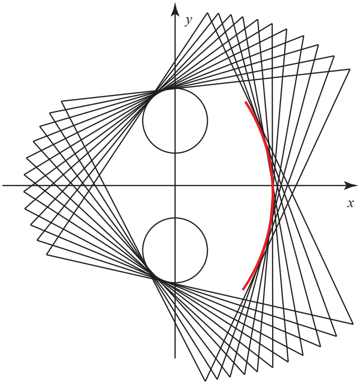
$$(x(p,Q(p)), y(p,Q(p)))$$

which depends on just one parameter p.

The envelope

It turns out that regardless of the circles or triangle involved, the envelope generated by the free side is always a circular arc.

The fact that the envelope exists for any value of the circle radius ρ means not only one rotor works, but there exists a family of rotors that will all work with the one trammel.



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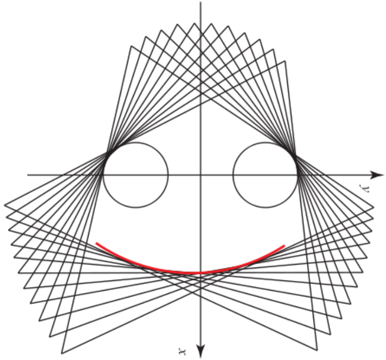
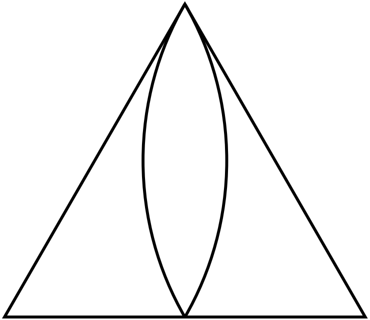
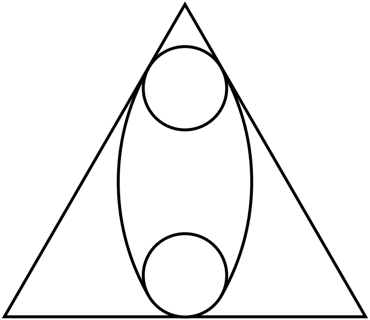


Example rotors



$\rho = 0$ leads to a two arc rotor



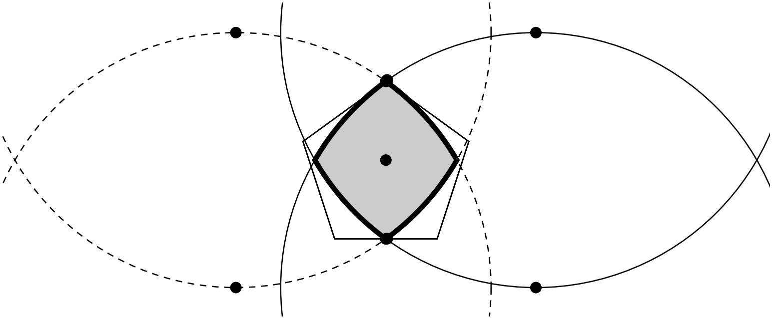
general rotor comprises four circular arcs

Drilling a perfect triangle

Higher odd n-gons

The same idea that works for the triangle also works for all odd n-gons.

In the case of the pentagon for $\rho = 0$, the rotor is the intersection of two special lenses called *vesicae piscis*.

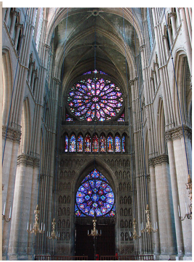


Vesica piscis

Definition: he intersection of two congruent circles where the centre of each circle lies on the circumference of the other.

The vesica piscis has wide ranging religious and mystical significance. Some examples include

- manuscripts
- carvings
- paintings
- coats of arms
- architecture



More sides

More sides are possible by adding more arcs.

However it turns out that only the first and last arc on each side are strictly necessary.

This allows us to simplify the rotor for higher odd n-gons to a truncated rotor.

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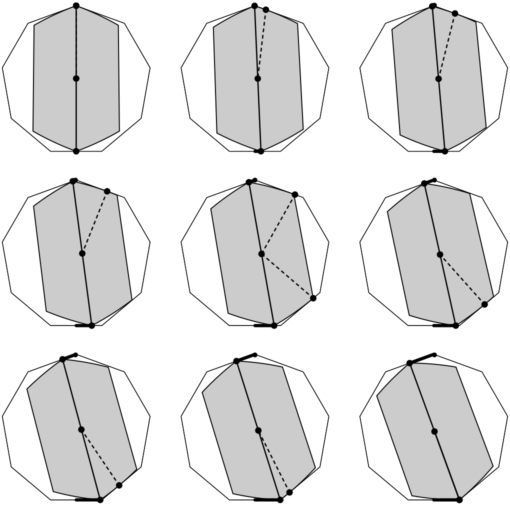
Truncated rotor for the nonagon



Acknowledgements

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  University of Birmingham
- Dan Velleman
  Amherst College

Where to find out more

- http://www.howround.com/
- Bryant and Sangwin, 2008, *How Round Is Your Circle?* Princeton University Press: New Jersey.
  UOA Library Call No: 620.00151 B9157h.
- http://demonstrations.wolfram.com/
- http://mathforum.org/wagon/
- Cox and Wagon, 2009 "Mechanical circle squaring," *College Mathematics Journal*, **40**, 238–247, doi:10.4169/193113409X458679.
- Cox and Wagon, 2011 "Drilling for polygons," *American Mathematical Monthly*, submitted.