At least four doors, numerous goats, a car, a frog, four lily pads and some probability

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Undergraduate Seminar

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Let’s make a deal

You are presented with three doors. Behind two of the doors there is a goat, whilst the remaining door has a car. The car is equally likely to be behind each door.
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Would you like to stick with your original choice of door, or change?
The number of students queued outside my office at the start of my consultation hour, $X$, is well approximated by a Poisson distribution with parameter 2:

$$\Pr(X = j) = e^{-2} \frac{2^j}{j!}, \quad j \in \{0, 1, 2, \ldots\}.$$
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What is the probability I open my door to find 2 students waiting, following a knock on my door, at the start of my consultation hour?
Let $A$ and $B$ be two events. Then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$
Consultation hour

Given I hear a knock on the door, I know that there must be at least 1 student waiting:

$$\Pr(X = j | X > 0) = \frac{\Pr(X = j \cap X > 0)}{\Pr(X > 0)}$$

$$= \frac{\Pr(X = j)}{1 - \Pr(X = 0)}, \quad j \in \{1, 2, \ldots\}.$$ 

So the probability I open my door to find 2 students waiting is

$$\frac{\Pr(X = 2)}{1 - \Pr(X = 0)} = \frac{2e^{-2}}{1 - e^{-2}} \approx 0.3130$$

(compare with $\Pr(X = 2) = 2e^{-2} \approx 0.2707$).
Let’s make a deal

Consider the possible allocations of goats/car to doors:

<table>
<thead>
<tr>
<th>Allocation</th>
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<th>Door 2</th>
<th>Door 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>Goat</td>
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</tr>
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Say you pick door 1 (any door will do!) – and let’s consider each scenario.
Let’s make a deal

Let $A$ and $B$ be two events. Then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$ 

Using this, we could evaluate

$$\Pr(H = h|C = c, S = s)$$

where $C$ be the RV corresponding to the door with the car, $S$ be the RV corresponding to the door you select, and $H$ be the RV corresponding to the door I open.
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where

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- $S$ is the RV corresponding to the door you select, and
- $H$ is the RV corresponding to the door I open.

But we want to know

$$\Pr(C = c | H = h, S = s).$$
Let’s make a deal

\[
\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}
\]

\[
= \frac{\Pr(A \cap B)}{\Pr(A \cap B) + \Pr(A \cap B')} 
\]

\[
= \frac{\Pr(A|B) \Pr(B)}{\Pr(A|B) \Pr(B) + \Pr(A|B') \Pr(B')}
\]

Generally, \(B_j, j = 1, \ldots, n,\)

\[
\Pr(B_j|A) = \frac{\Pr(A|B_j) \Pr(B_j)}{\sum_{i=1}^{n} \Pr(A|B_i) \Pr(B_i)}
\]
Let’s make a deal

\[
\Pr(C = c | H = h, S = s) = \frac{\Pr(H = h | S = s, C = c) \Pr(C = c)}{\sum_{j=1}^{3} \Pr(H = h | S = s, C = j) \Pr(C = j)}.
\]

So, for example,

\[
\Pr(C = 1 | H = 2, S = 1) = \frac{\Pr(H = 2 | S = 1, C = 1)}{\frac{1}{2}} = \frac{1}{3}
\]

and

\[
\Pr(C = 3 | H = 2, S = 1) = \frac{2}{3}.
\]
We wish to model the position of a frog on four lily pads after \( n \) days, \( X_n \in \{1, 2, 3, 4\} \) for \( n = \{0, 1, 2, \ldots\} \).
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We can form a matrix $P$, with entry $(i, j)$ telling us the probability of jumping from pad $i$ to pad $j$: for example,

$$P = \begin{pmatrix}
0 & 1/2 & 1/2 & 0 \\
1/3 & 0 & 1/3 & 1/3 \\
1/3 & 1/3 & 0 & 1/3 \\
0 & 1/2 & 1/2 & 0
\end{pmatrix}.$$
Discrete time Markov chains

So given
\[ p(n) = (\Pr(X_n = 1), \Pr(X_n = 2), \Pr(X_n = 3), \Pr(X_n = 4)) , \]
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The probability of the frog being on each pad remains unchanged after one (and hence any number of) step(s).
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We hence have a stationary distribution \( \pi \) such that

\[ \pi = \pi P. \]

The probability of the frog being on each pad remains unchanged after one (and hence any number of) step(s).

This distribution, \( \pi \), is equal to the limiting distribution (for certain DTMCs as considered here):

\[ \lim_{n \to \infty} p(n) = p(0) \lim_{n \to \infty} P^n = \pi. \]
For our friendly frog, we have

$$\pi = (0.2, 0.3, 0.3, 0.2)^T.$$
More goats... well, less goats and then more goats

Now we wish to model the number of female goats (henceforth called goats) in a paddock at the end of each breeding cycle, $X_t$, where $X_t \in \{0, 1, \ldots, N\}$ for $t = \{0, 1, \ldots\}$, and $N$ is the known maximum number of goat that can fit in the paddock.
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$$\Pr(X_{(t+1)^-} = n) = \binom{m}{m-n} d^{(m-n)} (1 - d)^n \quad (n \leq m).$$
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We can place these probabilities in a matrix $D$ (as we did for $P$ for the frog, before).
Now, if there are $X_{(t+1)^-} = n$ goats just before birth, there is $X_{(t+1)} = k$ goats following birth with probability

$$\Pr(X_{(t+1)} = k) = \binom{n}{k - n} b^{(k-n)} (1 - b)^{(2n-k)}$$

for $k = n, \ldots, \min\{2n, N - 1\}$, and if $n \geq N/2$,

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Then, the matrix $P$ for our model of goat dynamics, is:

$$P = DB,$$

the matrix product of our matrix $D$ and our matrix $B$. 
If we evaluate the stationary (limiting) distribution of our goat model, we find:

$$\pi = (1, 0, 0, \ldots, 0)$$

where $\pi$ is of length $N$.

In the long-term we will encounter a time at which we have no goats. Obviously, with no goats, we will remain with no goats forever.
No goats!

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But what happens if we look at realisations of this model...
Some goats, for a very long time...

**Figure:** 20 simulations with $N = 100$, $b = 0.4$ and $d = 0.25$. 
Some goats, for a very long time...

How can we describe (evaluate) this behaviour?
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We wish to evaluate the distribution of the number of goats, *conditional* on there being at least one goat in the paddock.
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We wish to evaluate the distribution of the number of goats, \textit{conditional} on there being at least one goat in the paddock.

So,

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\Pr(X_t = j | X_t > 0) = \frac{\Pr(X_t = j \cap X_t > 0)}{\Pr(X_t > 0)} = \frac{\Pr(X_t = j)}{1 - \Pr(X_t = 0)}, \quad j \in \{1, 2, \ldots, N\}.
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= \frac{\Pr(X_t = j)}{1 - \Pr(X_t = 0)}, \quad j \in \{1, 2, \ldots, N\}.
\]

This is a *conditional* distribution—conditioned on non-extinction.
Some goats, for a very long time...

We can now consider this distribution in the limit as $t \to \infty$. This is called a *limiting conditional* distribution.
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For our model, it is equal to the *quasi-stationary* distribution – an initial distribution which is invariant following one step and reconditioning on non-extinction.
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This distribution can be shown to equal:

$$\pi^* P^* = \rho \pi^*$$

where $P^*$ is the matrix $P$ with row and column corresponding to the \textit{absorbing state} 0 removed, and $\rho$ is the real, maximum-modulus eigenvalue of $P^*$ and $\pi^*$ is the left eigenvector (normalised to sum to 1) corresponding to $\rho$. 
Quasi-stationarity

Figure: Approximate QSD based upon simulations from $t = 500 \to 1000$. 
Pick a door, any door!
Which door?
More goats!

Research questions
Discrete time Markov chains

Quasi-stationarity

Figure: Exact QSD.
Quasi-stationarity

Figure: Relative error in QSD.
Current research

1. Using these types of methods to determine optimal strategies for prolonging species persistence.

2. Developing methods for approximating these distributions, to enable (efficient) evaluation.

3. Discovering results which allow us to determine if these distributions exist and if they are unique, and what they correspond to if not limiting or unique.

4. ...and much much more ...
Take home messages

1. Importance of conditioning on information.

2. Usefulness of probability, and in particular Markov chains, for (biological) modelling.

3. Many open research questions in this field.
Thanks for your attention!

http://www.maths.adelaide.edu.au/joshua.ross