

Add one part chaos, one part topology,
and stir well...

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Fluid flows are everywhere

Stirring and mixing of fluid occurs everywhere, from transport of heat, salt, pollutants etc. in the ocean and atmosphere, to making a cake or a cup of coffee, or mixing on a micro-scale in new laboratory-on-a-chip forensic equipment.

However, the underlying processes are often described by exactly the same mathematical techniques.

What is stirring and mixing?

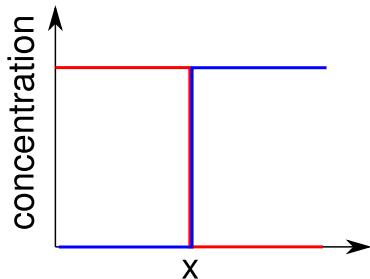
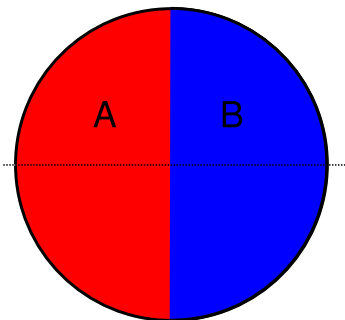
Mixing is the *passive* process by which molecular diffusion smooths out (or reduces) gradients in the concentration of a chemical.

Stirring is the *active* process of moving a fluid in order to generate (or increase) gradients in the concentration of a chemical.

So in some sense stirring is the opposite of mixing!

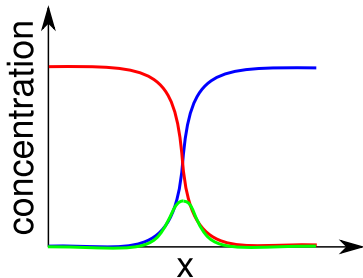
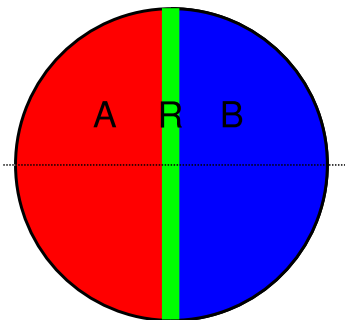
So why stir in the first place?

Consider two initially segregated chemicals A and B that we want to undergo the reaction $A + B \rightarrow R$.



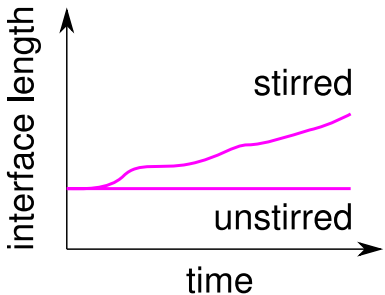
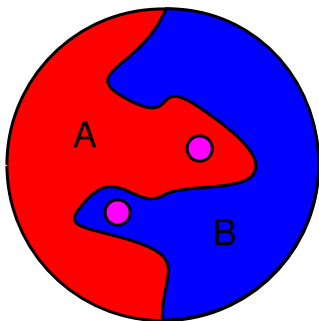
Without stirring reactions are slow

Without stirring the reaction occurs very slowly because we rely on molecular diffusion alone to bring A into contact with B .



Stirring promotes a faster reaction

Stirring stretches the interface between the chemicals and increases the area over which A and B may come into contact.



How do we follow stretching of the interface?

Stirring produces a fluid velocity field $(u(x, y, t), v(x, y, t))$ which satisfies some governing fluid equation (e.g. *Navier–Stokes*).

Suppose particles in the interface have position $(x(t), y(t))$.

If they move passively with the flow their trajectories satisfy

$$\begin{aligned}\frac{dx}{dt} &= u(x(t), y(t), t), \\ \frac{dy}{dt} &= v(x(t), y(t), t).\end{aligned}$$

What do we know about the fluid velocity?

The velocity components $u(x, y, t)$ and $v(x, y, t)$ are related because the fluid flow must conserve mass.

For an *incompressible* flow the velocity must have zero *divergence*.

This means that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Consequently, the velocity can be written as the *curl* of a *streamfunction* $\psi(x, y, t)$ according to

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Particle paths satisfy a Hamiltonian system

Particle paths $(x(t), y(t))$ satisfy a *Hamiltonian system* with one degree of freedom, where the Hamiltonian is the streamfunction.

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial\psi}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial\psi}{\partial x}\end{aligned}$$

So what are the properties of solutions to these equations?

Is it possible to stir badly?

Steady flows are bad!

The streamfunction is a *conserved quantity* on particle paths.

The Hamiltonian system for particle paths is *integrable*.

Particles are constrained to live on closed curves.

Nearby particles separate 'slowly' (*algebraically*).

The interface between chemicals stretches 'slowly'.

This is bad for our chemical reaction!

Is it possible to stir well?

A time-dependent flow is required.

Particle paths are generally not integrable...

Chaotic particle paths are possible!

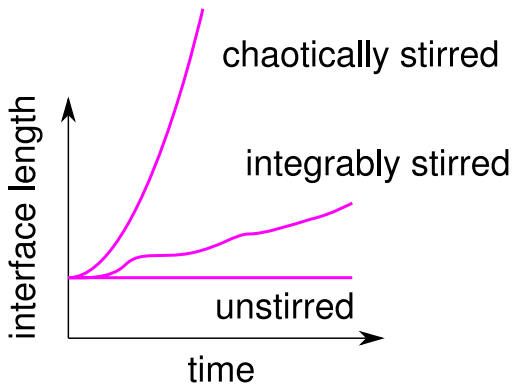
Solutions are sensitive to initial conditions.

Nearby particles separate 'quickly' (*exponentially*).

The interface between chemicals stretches 'quickly'.

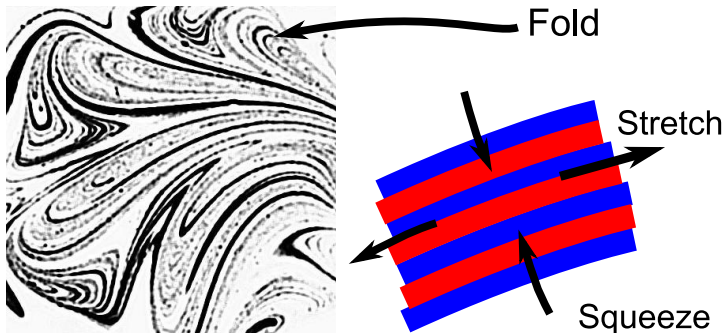
This is good for our chemical reaction.

No stirring, bad stirring, good stirring



Stretching results in squeezing and folding

Exponential stretching must lead to folding of the interface, and the formation of narrow alternating chemical striations.



Striations quickly become thin enough that even a small amount of molecular diffusion brings the chemicals together.

So how do you generate chaos?

The good news is that chaos is quite a generic phenomenon and so is easy to create...

...the bad news is that the 'amount' of chaos can depend sensitively on the properties of the fluid, the details of the geometry, and so on.

So is there any way to *guarantee* some chaotic dynamics regardless of the fluid properties and exact details of geometry?

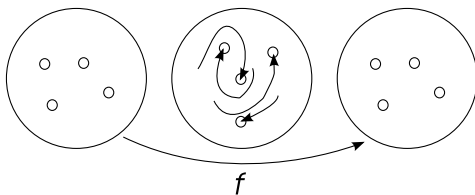
How can you guarantee good stirring?

Consider time-periodic flow driven by n stirrers.

$$u(x, y, t) = u(x, y, t + T)$$

$$v(x, y, t) = v(x, y, t + T)$$

The fluid domain is the n -punctured disk, and the flow induces a continuous map f of the domain onto itself.



Can you move the stirrers in such a way that f is guaranteed to have chaotic dynamics?

The Thurston–Nielsen classification theorem

The map f belongs to one of three categories:

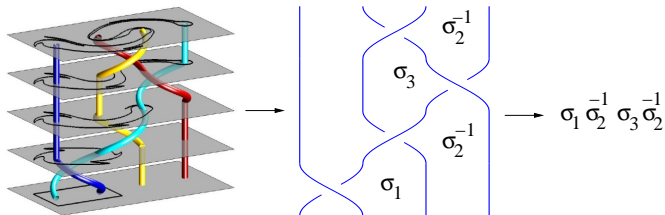
Finite-order: f is topologically trivial. Some iterate of f is topologically the same as the *identity map*, and moves the fluid in a way that can be untangled while holding the stirrers fixed.

Pseudo-Anosov: f is topologically non-trivial. It moves the fluid in a way that cannot be untangled, stretches exponentially (almost) everywhere and is *guaranteed* to have chaotic dynamics.

Reducible: f leaves some curves invariant that divide the domain into smaller regions of type 1 and 2.

How do you characterise flow topology?

The classes that f can belong to are labelled by *braids* describing the topology of the stirrer motion.



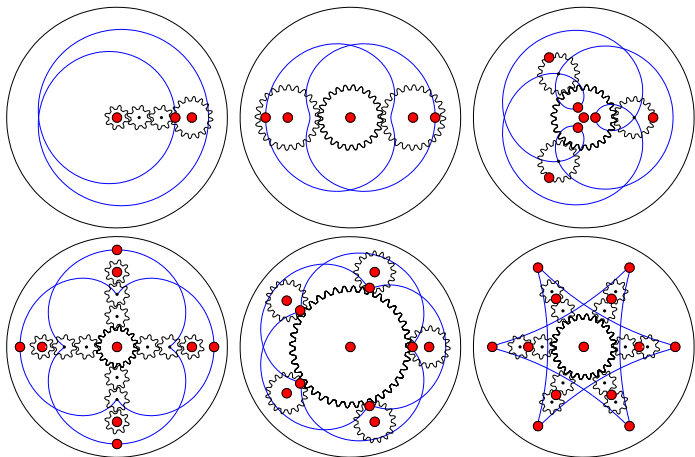
You need at least three stirrers!

Any flow driven by one or two stirrers is topologically trivial.

With three or more stirrers you can generate a non-trivial topology.

This is best illustrated with pipe cleaners!

All this can be done with simple machinery



Summary

Why stir?

Stirring increases the interfacial area between two chemicals, enabling mixing by molecular diffusion to enhance the rate of reaction.

What is good stirring?

A flow which generates chaotic particle paths causes fast stretching and folding, which quickly forms striations of chemicals thin enough for molecular diffusion to homogenise.

How can you guarantee chaotic dynamics?

Use three or more stirrers and braid them to induce a topologically non-trivial flow.