Holomorphic null curves and the conformal Calabi-Yau problem

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Plan of the talk

- Basics on minimal surfaces
- Connection with holomorphic null curves in $\mathbb{C}^3$
- Our contribution to the Calabi-Yau problem
- A brief history of the subject from Calabi’s conjecture 1965
- The main tools: Riemann-Hilbert problem for null curves, exposing points, gluing techniques
- Proper null curves in $\mathbb{C}^3$ with a bounded coordinate function
- Applications to null curves in $SL_2(\mathbb{C})$ and to Bryant surfaces in the hyperbolic 3-space

Based on joint work with Antonio Alarcón, University of Granada.
The only area minimizing surfaces of rotation are planes and catenoids.
Lagrange Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then a smooth graph $(x, y, f(x, y)) \subset \overline{\Omega} \times \mathbb{R}$ is a critical point of the area functional with prescribed boundary values iff

\[(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0;\]
equivalently,

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0.
\]

This is known as the equation of minimal graphs.
A smoothly immersed surface $M \rightarrow \mathbb{R}^3$ is said to be \textbf{minimal} if its mean curvature is identically zero.
The helicoid (Archimedes’ screw)

1776 **Meusnier** The helicoid is a minimal surface.

\[ x = \rho \cos(\alpha \theta), \quad y = \rho \sin(\alpha \theta), \quad z = \theta \]

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in \( \mathbb{R}^3 \).
A relative of helicoid - Dini’s surface

A surface with constant negative curvature, named after Ulisse Dini (1845 – 1918), an Italian mathematician and politician born in Pisa.
1873 **Plateau** Minimal surfaces can be obtained as soap films.

1932 **Douglas, Radó** Every continuous injective closed curve in $\mathbb{R}^3$ spans a minimal surface.
1865  **Riemann** On the way to this solution, Riemann and others discovered new examples of minimal surfaces using the Weierstrass representation.
Conformal minimal surfaces in $\mathbb{R}^3$

Assume that $M$ is a **Riemann surface**, i.e., a smooth orientable surface with a choice of a conformal complex structure.

**Definition**

A smooth immersion $M \to \mathbb{R}^3$ is **conformal** if it preserves angles, and is **minimal** if its mean curvature is identically zero.

- Every Riemann surface is conformally equivalent to a closed embedded surface in $\mathbb{R}^3$ (Rüedy 1971).
- Denote by $\Theta : M \to \mathbb{R}$ its mean curvature and by $\nu : M \to S^2 \subset \mathbb{R}^3$ its Gauss map. Then
  \[ \Delta G = 2\Theta \nu. \]
- Hence a conformal immersion $M \to \mathbb{R}^3$ is minimal iff it is harmonic.
Complete bounded minimal surfaces in $\mathbb{R}^3$

- An immersion $M \to \mathbb{R}^3$ is said to be **complete** if the pullback of the Euclidean metric on $\mathbb{R}^3$ is a complete metric on $M$. Equivalently, the image of any divergent curve in $M$ has infinite length.

- We give a contribution to the **conformal Calabi-Yau problem**:

**Theorem**

*Every bordered Riemann surface admits a complete conformal minimal (=harmonic) immersion into $\mathbb{R}^3$ with bounded image.*

- What is new in comparison to all existing results is that we do not change the complex structure on the Riemann surface.

Holomorphic null curves in $\mathbb{C}^3$

This theorem is a corollary to a comparable result concerning \textit{holomorphic null curves} in $\mathbb{C}^3$.

\begin{definition}{Null curves}

Let $M$ be a Riemann surface. A holomorphic immersion

$$F = (F_1, F_2, F_3) : M \to \mathbb{C}^3$$

is a \textbf{null curve} if the derivative $F' = (F'_1, F'_2, F'_3)$ with respect to any local holomorphic coordinate $\zeta = x + iy$ on $M$ satisfies

$$(F'_1)^2 + (F'_2)^2 + (F'_3)^2 = 0.$$  

\end{definition}
If $F = G + iH : M \to \mathbb{C}^3$ is a holomorphic null curve, then
\[
G = \Re F : M \to \mathbb{R}^3, \quad H = \Im F : M \to \mathbb{R}^3
\]
are conformal harmonic (hence minimal) immersions into $\mathbb{R}^3$.

Conversely, a conformal minimal immersion $G : \mathbb{D} \to \mathbb{R}^3$ of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the real part of a holomorphic null curve $F : \mathbb{D} \to \mathbb{C}^3$. This fails on non-simply connected Riemann surfaces due to the period problem for the harmonic conjugate.

If $F = G + iH : M \to \mathbb{C}^3$ is a null curve then
\[
F^* ds^2_{\mathbb{C}^3} = 2G^* ds^2_{\mathbb{R}^3} = 2H^* ds^2_{\mathbb{R}^3}.
\]

It follows that the real and the imaginary part of a complete null curve in $\mathbb{C}^3$ are complete conformal minimal surfaces in $\mathbb{R}^3$.
Let \( F = G + \imath H = (F^1, F^2, F^3) : M \to \mathbb{C}^3 \) be a holomorphic null curve and \( \zeta = x + \imath y \) a local holomorphic coordinate on \( M \). Then

\[
0 = \sum_{j=1}^{3} (F^j_\zeta)^2 = \sum_{j=1}^{3} (F^j_x)^2 = \sum_{j=1}^{3} \left( G^j_x + \imath H^j_x \right)^2
\]

\[
= \sum_{j=1}^{3} \left( (G^j_x)^2 - (H^j_x)^2 \right) + 2\imath \sum_{j=1}^{3} G^j_x H^j_x.
\]

Since \( H_x = -G_y \) by the CR equations, this reads

\[
0 = |G_x|^2 - |G_y|^2 - 2\imath G_x \cdot G_y \iff |G_x| = |G_y|, \ G_x \cdot G_y = 0.
\]

It follows that \( G \) is conformal harmonic and

\[
F^* ds^2_{\mathbb{C}^3} = |F_x|^2 (dx^2 + dy^2) = 2|G_x|^2 (dx^2 + dy^2) = 2G^* ds^2_{\mathbb{R}^3} = 2H^* ds^2_{\mathbb{R}^3}.
\]
Example: The catenoid and the helicoid are conjugate minimal surfaces – the real and the imaginary part of the same null curve

\[ F(\zeta) = (\cos \zeta, \sin \zeta, -i \zeta), \quad \zeta = x + iy \in \mathbb{C}. \]

Consider the family of minimal surfaces \((t \in \mathbb{R})\):

\[ G_t(\zeta) = \Re \left( e^{it} F(\zeta) \right) \]

\[ = \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix} \]

At \( t = 0 \) we have a catenoid, and at \( t = \pm \pi/2 \) we have a (left or right handed) helicoid.
The family of minimal surfaces $G_t(ζ) = \Re(e^{it}F(ζ))$, $t \in \mathbb{R}$:
The first main result

This shows that the existence of complete bounded conformal minimal immersions $M \to \mathbb{R}^3$ follows from part (B) of the following result.

**Theorem**

Let $M$ be a bordered Riemann surface.

(A) There exists a proper complete holomorphic immersion $M \to \mathbb{B}^2$ into the unit ball of $\mathbb{C}^2$.

(B) There exists a proper complete null holomorphic embedding $F : M \hookrightarrow \mathbb{B}^3$ into the unit ball of $\mathbb{C}^3$.

(B) answers a question of Martín, Umehara and Yamada (2009).

Part (A) holds for immersions into any Stein manifold $(X, ds^2)$ of dimension $> 1$ with a chosen Riemannian metric.

[A. Alarcón, F. Forstnerič: Every bordered Riemann surface is a complete proper curve in a ball. Math. Ann. 2013]
1985 **Løw** Every strongly pseudoconvex Stein domain $M$ admits a proper holomorphic embedding $\phi : M \to \mathbb{D}^m$ into a polydisc.

Let $h : \mathbb{D} \to \mathbb{B}^2$ be a complete proper holomorphic immersion. Then

$$H : \mathbb{D}^m \to (\mathbb{B}^2)^m \subset \mathbb{C}^{2m}, \quad H(z_1, \ldots, z_m) = (h(z_1), \ldots, h(z_m))$$

is a complete proper holomorphic immersion. Similarly we get complete proper holomorphic embeddings $\mathbb{D}^m \to (\mathbb{B}^3)^m$.

Hence $F = H \circ \phi : M \to (\mathbb{B}^2)^m$ is a complete proper immersion.

**Corollary**

*Every strongly pseudoconvex Stein domain admits a complete bounded holomorphic embedding into $\mathbb{C}^N$ for large $N$.***
A brief history of the Calabi-Yau problem

1965  **E. Calabi** conjectured that there does not exist any complete minimal surface in $\mathbb{R}^3$ with a bounded coordinate function.

1977  **P. Yang** asked whether there exist any complete bounded complex submanifolds of $\mathbb{C}^n$ for $n > 1$.

1979  **P. Jones** constructed a complete bounded holomorphic immersion $\mathbb{D} \rightarrow \mathbb{C}^2$ of the disc, using BMO methods.

1980  **L.P. Jorge & F. Xavier** constructed complete minimal surfaces in $\mathbb{R}^3$ with a bounded coordinate, disproving Calabi’s conjecture.

1996  **N. Nadirashvili** constructed a complete bounded conformal minimal immersion $\mathbb{D} \rightarrow \mathbb{R}^3$, hence a complete null curve in $\mathbb{C}^3$. His technique cannot be refined to control the imaginary part.
A brief history...continued

2000  **S.-T. Yau: Review of geometry and analysis** ("The Millenium Lecture"). Mathematics: frontiers and perspectives, AMS. The problem became known as the **Calabi-Yau problem**.

2008  **T.H. Colding and W.P. Minicozzi II**: An embedded complete minimal surface $M \hookrightarrow \mathbb{R}^3$ with finite genus and at most countably many ends is proper in $\mathbb{R}^3$, and $M$ is algebraic.

2009  **F. Martín, M. Umehara and K. Yamada** constructed complete bounded holomorphic curves in $\mathbb{C}^2$ with arbitrary finite topology.

2012  **L. Ferrer, F. Martín and W.H. Meeks** found complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topology.

2013  **A. Alarcón and F.J. Lopez**: Examples of (i) complete bounded null curves in $\mathbb{C}^3$, (ii) complete bounded immersed holomorphic curves in $\mathbb{C}^2$ with arbitrary topology, and (iii) complete bounded **embedded** holomorphic curves in $\mathbb{C}^2$. 
2014  **J. Globevnik:** For any $n > 1$ there exists a holomorphic function $f$ on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ such that every level set $f = c$ is complete (i.e., divergent curves in $\{f = c\}$ have infinite length).

Since most such hypersurfaces are smooth, this gives an optimal answer to the question of Paul Yang from 1977.

**Question:** What is the possible topology and complex structure on such hypersurfaces?
Geometry of the null quadric

- The **directional variety** of null curves:

  \[ A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0 \} \]

- \( A \) is a complex cone with vertex at 0; \( A^* = A \setminus \{0\} \) is smooth.

- \( L = \{ [z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1^2 + z_2^2 + z_3^2 = 0 \} \cong \mathbb{CP}^1 \rightarrow \mathbb{CP}^2. \)

- \( pr : A^* \rightarrow L \) is a holomorphic fiber bundle with fiber \( \mathbb{C}^*. \)

- It follows that \( A^* \) is an **Oka manifold**.

- The spinor representation:

  \[ \pi : \mathbb{C}^2 \rightarrow A, \quad \pi(u, v) = (u^2 - v^2, i(u^2 + v^2), 2uv). \]

  The map \( \pi : \mathbb{C}^2 \setminus \{0\} \rightarrow A^* \) is a nonramified two-sheeted covering.
Construction of holomorphic null curves

Let $M$ be a bordered Riemann surface. Fix a nowhere vanishing holomorphic 1-form $\theta$ on $M$; such exists by the Oka-Grauert principle. There is a bijective correspondence (up to constants)

$$\{ F : M \to \mathbb{C}^3 \text{ null curve} \} \leftrightarrow \{ f : M \to A^* \text{ holomorphic, } f \theta \text{ exact} \}$$

$$F(x) = F(p) + \int_p^x f \theta, \quad dF = f \theta.$$ 

**Theorem (The Oka principle for null curves)**

Every continuous map $f_0 : M \to A^*$ of an open Riemann surface $M$ to $A^*$ is homotopic to a holomorphic map $f : M \to A^*$ such that $f \theta$ has vanishing periods. Furthermore, a generic null curve is an embedding. The same holds whenever $A^* \subset \mathbb{C}^n$, $n \geq 3$, is an Oka manifold.

Idea of the construction of complete bounded holomorphic immersions - Pythagora’s theorem

Let $F_0 : \overline{M} \rightarrow \mathbb{C}^n$ be a holomorphic immersion satisfying $|F_0| \geq r_0 > 0$ on $bM$. We try to increase the boundary distance on $M$ with respect to the induced metric by a fixed number $\delta > 0$.

To this end, we approximate $F_0$ uniformly on a compact set in $M$ by an immersion $F_1 : \overline{M} \rightarrow \mathbb{C}^n$ which at a point $x \in bM$ adds a displacement for approximately $\delta$ in a direction $V \in \mathbb{C}^n$, $|V| = 1$, approximately orthogonal to the point $F_0(x) \in \mathbb{C}^n$. The boundary distance increases by $\approx \delta$, while the outer radius increases to

$$|F_1(x)| \approx \sqrt{|F_0(x)|^2 + \delta^2} \approx |F_0(x)| + \frac{\delta^2}{2|F_0(x)|} \leq |F_0(x)| + \frac{\delta^2}{2r_0}.$$ 

By choosing a sequence $\delta_j > 0$ such that $\sum_j \delta_j = +\infty$ while $\sum_j \delta_j^2 < \infty$, we obtain by induction a limit immersion $F : M \rightarrow \mathbb{C}^n$ with bounded outer radius and with complete metric $F^* ds^2$. 
The main tools

- This idea can be realized on short arcs \( I \subset bM \), on which \( F_0 \) does not vary too much, by solving a **Riemann-Hilbert problem**.

- Globally this method alone could lead to ‘sliding curtains’, creating shortcuts in the new induced metric on \( M \).

- **To localize the problem** and **eliminate any shortcuts**, we subdivide \( bM = \bigcup_j I_j \) into a finite union of short arcs such that two adjacent arcs \( I_{j-1}, I_j \) meet at a common endpoint \( x_j \). At the point \( p_j = F(x_j) \in \mathbb{C}^n \) we attach to \( F_0(M) \) a smooth real curve \( \lambda_j \) of length \( \delta \) whose other endpoint \( q_j \) increases the outer radius by \( \delta^2 \).

- By the method of **exposing boundary points** we modify the immersion so that \( F_0(x_j) = q_j \). Hence any curve in \( M \) terminating on \( bM \) near \( x_j \) is elongated by approximately \( \delta > 0 \).

- In the next step we use a Riemann-Hilbert problem to increase the boundary distance on the arcs \( I_j \) by approximately \( \delta \). These local modifications are glued together by the method of sprays.
Theorem (Riemann-Hilbert problem for null discs)

Let \( F_0 : \overline{D} \to \mathbb{C}^3 \) be a null holomorphic immersion, let \( V \in A^* \), let \( \mu : bD \to [0, +\infty) \) be a continuous function, and consider the map

\[
Y : bD \times \overline{D} \to \mathbb{C}^3, \quad Y(\zeta, z) = F_0(\zeta) + \mu(\zeta)zV.
\]

Given numbers \( \varepsilon > 0 \) and \( 0 < r < 1 \), there exist a number \( r' \in [r, 1) \) and a null holomorphic immersion \( F : \overline{D} \to \mathbb{C}^3 \) satisfying the following:

- \( \text{dist}(F(\zeta), Y(\zeta, bD)) < \varepsilon \) for \( \zeta \in bD \).
- \( \text{dist}(F(\rho \zeta), Y(\zeta, \overline{D})) < \varepsilon \) for \( \zeta \in bD \) and \( \rho \in [r', 1) \).
- \( F \) is \( \varepsilon \)-close to \( F_0 \) in the \( C^1 \) topology on \( \{ \zeta \in \mathbb{C} : |\zeta| \leq r' \} \).

Furthermore, if \( J \) is a compact arc in \( bD \) such that \( \mu \) vanishes on \( bD \setminus J \), and \( U \) is an open neighborhood of \( J \) in \( \overline{D} \), then

- one can choose \( F \) to be \( \varepsilon \)-close to \( F_0 \) in the \( C^1 \) topology on \( \overline{D} \setminus U \).
Proof

Consider the unbranched two-sheeted holomorphic covering

$$\pi: \mathbb{C}^2 \setminus \{(0,0)\} \rightarrow A^*, \quad \pi(u, v) = (u^2 - v^2, i(u^2 + v^2), 2uv).$$

Since $\overline{D}$ is simply connected, the map $F'_0: \overline{D} \rightarrow A^*$ lifts to a map $(u, v): \overline{D} \rightarrow \mathbb{C}^2 \setminus \{(0,0)\}$. Hence we have

$$F'_0 = \pi(u, v) = \left(u^2 - v^2, i(u^2 + v^2), 2uv\right) \in A^*$$
$$V = \pi(a, b) = \left(a^2 - b^2, i(a^2 + b^2), 2ab\right) \in A^*$$
$$\eta = \sqrt{\mu}: b\overline{D} \rightarrow [0, \infty)$$
$$\eta(\zeta) \approx \tilde{\eta}(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \quad \text{(rational approximation)}$$
$$\mu(\zeta) \approx \tilde{\eta}^2(\zeta) = \sum_{j=1}^{2N} B_j \zeta^{j-2m}.$$
For any integer $n \in \mathbb{N}$ we consider the following functions and maps

$$u_n(\xi) = u(\xi) + \sqrt{2n+1} \tilde{\eta}(\xi) \xi^n a,$$
$$v_n(\xi) = v(\xi) + \sqrt{2n+1} \tilde{\eta}(\xi) \xi^n b,$$
$$\Phi_n(\xi) = \pi(u_n(\xi), v_n(\xi)) = (u_n^2 - v_n^2, i(u_n^2 - v_n^2), 2u_nv_n) : \overline{D} \to A^*,$$
$$F_n(\zeta) = F_0(0) + \int_0^\zeta \Phi_n(\xi) \, d\xi, \quad \zeta \in \overline{D}.$$  

Then $F_n : \overline{D} \to \mathbb{C}^3$ is a null disc of the form

$$F_n(\zeta) = F_0(\zeta) + B_n(\zeta) \, V + A_n(\zeta).$$
The \( \mathbb{C} \)-valued term \( B_n \) equals

\[
B_n(\zeta) = (2n+1) \sum_{j=1}^{2N} \int_0^\zeta B_j \zeta^{2n+j-2m} d\zeta
\]

\[
= \sum_{j=1}^{2N} \frac{2n+1}{2n+1+j-2m} B_j \zeta^{2n+1+j-2m}.
\]

Since the coefficients \( (2n+1)/(2n+1+j-2m) \) in the sum for \( B_n \) converge to 1 as \( n \to +\infty \), we have

\[
\sup_{|\zeta| \leq 1} \left| B_n(\zeta) - \zeta^{2n+1} \tilde{\eta}^2(\zeta) \right| \to 0 \quad \text{as} \quad n \to \infty.
\]
Proof—continued

The remainder $\mathbb{C}^3$-valued term $A_n(\zeta)$ equals

$$A_n(\zeta) = 2\sqrt{2n+1} \int_0^\zeta \sum_{j=1}^N A_j \xi^{n+j-m}(u(\xi)(a,ia,b) + v(\xi)(-b,ib,a)) d\xi$$

$$|A_n(\zeta)| \leq 2\sqrt{2n+1} C_0 \sum_{j=1}^N |A_j| \int_0^{|\zeta|} |\xi|^{n+j-m} d|\xi|$$

$$\leq 2C_0 \sum_{j=1}^N \frac{\sqrt{2n+1}}{n+1+j-m} |A_j|.$$ 

It follows that $|A_n| \to 0$ uniformly on $\overline{D}$ as $n \to +\infty$. Hence

$$F_n(\zeta) \approx F_0(\zeta) + \zeta^{2n+1} \tilde{\mu}(\zeta) V, \quad \zeta \in \overline{D}.$$ 

The theorem follows from this estimate.
Null curves with a bounded coordinate

The Riemann-Hilbert problem for null curves also gives the following.

**Theorem**

*Every bordered Riemann surface* $M$ *carries a proper holomorphic null embedding* $F = (F_1, F_2, F_3) : M \to \mathbb{C}^3$ *such that the function* $F_3$ *is bounded on* $M$. *(Thus* $(F_1, F_2) : M \to \mathbb{C}^2$ *is a proper map.)*

- This contrasts the theorem of **Hoffman and Meeks** (1990) that the only properly immersed minimal surfaces in $\mathbb{R}^3$ contained in a half-space are planes.

- This result has a nontrivial line of corollaries. A null curve in $SL_2(\mathbb{C})$ is a holomorphic immersion $F : M \to SL_2(\mathbb{C})$ of an open Riemann surface $M$ which is directed by the variety

$$\mathcal{B} = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : \det z = z_{11}z_{22} - z_{12}z_{21} = 0 \right\} \subset \mathbb{C}^4.$$
Null curves in $SL_2(\mathbb{C})$

- The biholomorphic map $\mathcal{T} : \mathbb{C}^3 \setminus \{z_3 = 0\} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$,

  \[ \mathcal{T}(z_1, z_2, z_3) = \frac{1}{z_3} \begin{pmatrix} 1 & z_1 + iz_2 \\ z_1 - iz_2 & z_1^2 + z_2^2 + z_3^2 \end{pmatrix}, \]

  carries null curves into null curves.

- Furthermore, if $F = (F_1, F_2, F_3) : M \to \mathbb{C}^3$ is a proper null curve such that $1/2 < |F_3| < 1$ on $M$, then $G = \mathcal{T} \circ F : M \to SL_2(\mathbb{C})$ is a proper null curve in $SL_2(\mathbb{C})$. This proves the following.

**Corollary**

*Every bordered Riemann surface carries a proper holomorphic null embedding into $SL_2(\mathbb{C})$.*
Bryant surfaces in hyperbolic 3-space

- The projection of a null curve in $SL_2(\mathbb{C})$ to the hyperbolic 3-space $\mathcal{H}^3 = SL_2(\mathbb{C})/SU(2)$ is a **Bryant surface**, i.e., a conformally immersed surface with constant mean curvature one in $\mathcal{H}^3$.

**Corollary**

*Every bordered Riemann surface is conformally equivalent to a properly immersed Bryant surface in the hyperbolic 3-space $\mathcal{H}^3$.***

2002 **Collin-Hauswirth-Rosenberg**  Properly *embedded* Bryant surfaces in $\mathcal{H}^3$ of finite topology have finite total curvature and regular ends. Hence our examples cannot be embedded.

- To the best of our knowledge, **these are the first examples of proper null curves in $SL_2(\mathbb{C})$, and Bryant surfaces in $\mathcal{H}^3$, with finite topology and hyperbolic conformal structure.**
A few open problems

- Does there exist a complete bounded holomorphic embedding $\mathbb{D} \hookrightarrow \mathbb{C}^2$ of the disc? Of an arbitrary bordered Riemann surface?

- Does there exist a proper minimal conformal immersion $M \hookrightarrow \mathbb{B}^3$ of an arbitrary bordered Riemann surface $M$?

- Is it possible to immerse or embed the ball $\mathbb{B}^2 \subset \mathbb{C}^2$ as a complete bounded complex submanifold of $\mathbb{C}^3$, $\mathbb{C}^4$, ...

- **Conjecture** (well known, likely very difficult): An orientable surface of finite topology with genus $g$ and $m$ ends properly embeds in $\mathbb{R}^3$ as a minimal surface if and only if $m \leq g + 2$.

- **Calabi’s conjecture** is still open in dimensions $n > 3$: Do there exist complete bounded minimal hypersurfaces of $\mathbb{R}^n$ when $n > 3$?