An Algebraic Approach to Internet Routing Day 2

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Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and path $p = i_1, i_2, i_3, \cdots, i_k$.

Semiring Path Weight

Weight function $w : E \to S$

$$w(p) = w(i_1,\ i_2) \otimes w(i_2,\ i_3) \otimes \cdots \otimes w(i_{k-1},\ i_k).$$

How about functions on arcs?

Weight function $w : E \to (S \to S)$

$$w(p) = w(i_1,\ i_2)(w(i_2,\ i_3)(\cdots w(i_{k-1},\ i_k)(a)\cdots)),$$

where a is some value originated by node i_k

How can we make this work?

Algebra of Monoid Endomorphisms ([GM08])

A homomorphism is a function that preserves structure. An endomprhism is a homomorphism mapping a structure to itself.

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \to S, \overline{0}, i, \omega)$ is a algebra of monoid endomorphisms (AME) if

- $\forall f \in F \ \forall b, c \in S : f(b \oplus c) = f(b) \oplus f(c)$
- $\forall f \in F : f(\overline{0}) = \overline{0}$
- $\exists i \in F \ \forall a \in S : i(a) = a$
- $\exists \omega \in F \ \forall a \in S : \omega(a) = \overline{0}$

Solving (some) equations over a AMEs

We will be interested in solving for x equations of the form

$$x = f(x) \oplus b$$

Let

$$\begin{array}{rcl} f^0 & = & i \\ f^{k+1} & = & f \circ f^k \end{array}$$

and

$$\begin{array}{rcl} f^{(k)}(b) & = & f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^k(b) \\ f^{(*)}(b) & = & f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^k(b) \oplus \cdots \end{array}$$

Definition (q stability)

If there exists a q such that for all b $f^{(q)}(b) = f^{(q+1)}(b)$, then f is *q-stable*. Therefore, $f^{(*)}(b) = f^{(q)}(b)$.

Key result (again)

Lemma

If f is q-stable, then $x = f^{(*)}(b)$ solves the AME equation

$$x = f(x) \oplus b.$$

Proof: Substitute $f^{(*)}(b)$ for x to obtain

$$\begin{aligned}
& f(f^{(*)}(b)) \oplus b \\
= \; & f(f^{(q)}(b)) \oplus b \\
= \; & f(f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^q(b)) \oplus b \\
= \; & f^1(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^{q+1}(b) \oplus b \\
= \; & f^0(b) \oplus f^1(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^{q+1}(b) \\
= \; & f^{(q+1)}(b) \\
= \; & f^{(q)}(b) \\
= \; & f^{(*)}(b)
\end{aligned}$$

AME of Matrices

Given an AME $S = (S, \oplus, F)$, define the semiring of $n \times n$-matrices over S,

$$\mathbb{M}_n(S) = (\mathbb{M}_n(S), \oplus, G),$$

where for $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(S)$ we have

$$(\mathbf{A} \oplus \mathbf{B})(i, j) = \mathbf{A}(i, j) \oplus \mathbf{B}(i, j).$$

Elements of the set G are represented by $n \times n$ matrices of functions in F. That is, each function in G is represented by a matrix $\mathbf{A}$ with $\mathbf{A}(i, j) \in F$. If $\mathbf{B} \in \mathbb{M}_n(S)$ then define $\mathbf{A}(\mathbf{B})$ so that

$$(\mathbf{A}(\mathbf{B}))(i, j) = \sum_{1 \le q \le n}^{\oplus} \mathbf{A}(i, q)(\mathbf{B}(q, j)).$$

Here we go again...

Path Weight

For graph $G = (V,\ E)$ with $w : E \to F$
The *weight* of a path $p = i_1, i_2, i_3, \cdots, i_k$ is then calculated as

$$w(p) = w(i_1,\ i_2)(w(i_2,\ i_3)(\cdots w(i_{k-1},\ i_k)(\omega_{\oplus})\cdots)).$$

adjacency matrix

$$\mathbf{A}(i,\ j) = \begin{cases} w(i,\ j) & \text{if } (i,\ j) \in E, \\ \omega & \text{otherwise} \end{cases}$$

We want to solve equations like these

$$\mathbf{X} = \mathbf{A}(\mathbf{X}) \oplus \mathbf{B}$$

Why do we need Monoid Endomorphisms??

Monoid Endomorphisms can be viewed as semirings

Suppose $(S, \oplus, F)$ is a monoid of endomorphisms. We can turn it into a semiring

$$(F, \hat{\oplus}, \circ)$$

where $(f \mathbin{\hat{\oplus}} g)(a) = f(a) \oplus g(a)$

Functions are hard to work with....

- All algorithms need to check equality over elements of semiring,
- $f = g$ means $\forall a \in S : f(a) = g(a)$,
- S can be very large, or infinite.

Convolution Product [GM08]

$(S, \oplus, \otimes, \overline{0}, \overline{1})$	a semiring
$(T, \bullet, \overline{1}_T)$	a monoid
$F \subseteq T \to S$	(suitably closed)

Construct a semiring $(F, \hat{\oplus}, \star)$

$$
\begin{aligned}
(f \mathbin{\hat{\oplus}} g)(a) &= f(a) \oplus g(a) \\
(f \star g)(a) &= \bigoplus_{a = b \bullet c} f(b) \otimes g(c)
\end{aligned}
$$

Note : when S is a ring and T is a commutative semigroup, this construction results in a ring called a commutative semigroup ring (R. Gilmer, 1984). Thanks to Snigdhayan Mahanta for pointing this out.

Lexicographic product of AMEs

$$(S,\ \oplus_S,\ F)\ \vec{\times}\ (T,\ \oplus_T,\ G) = (S \times T,\ \oplus_S\ \vec{\times}\ \oplus_T,\ F \times G)$$

Theorem ([Sai70, GG07, Gur08])

$$\mathrm{D}(S\ \vec{\times}\ T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(S) \vee \mathrm{K}(T))$$

Where

Property	Definition
D	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$
C	$\forall a, b, f : f(a) = f(b) \implies a = b$
K	$\forall a, b, f : f(a) = f(b)$

Functional Union of AMEs

$$(S,\ \oplus,\ F) +_{\mathrm{m}} (S,\ \oplus,\ G) = (S,\ \oplus,\ F + G)$$

Fact

$$\mathrm{D}(S +_{\mathrm{m}} T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T)$$

Where

Property	Definition
D	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$

Left and Right

right

$$\textbf{right}(S, \oplus, F) = (S, \oplus, \{i\})$$

left

$$\textbf{left}(S, \oplus, F) = (S, \oplus, K(S))$$

where $K(S)$ represents all constant functions over S. For $a \in S$, define the function $\kappa_a(b) = a$. Then $K(S) = \{\kappa_a \mid a \in S\}$.

Facts

The following are always true.

$\text{D}(\textbf{right}(S))$
$\text{D}(\textbf{left}(S))$ (assuming $\oplus$ is idempotent)
$\text{C}(\textbf{right}(S))$
$\text{K}(\textbf{left}(S))$

Scoped Product

$$S \Theta T = (S \vec{\times} \mathbf{left}(T)) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)$$

Theorem

$$\mathrm{D}(S \Theta T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T).$$

Proof.

$$\begin{aligned}
& \mathrm{D}(S \Theta T) \\
& \mathrm{D}((S \vec{\times} \mathbf{left}(T)) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)) \\
\iff & \mathrm{D}(S \vec{\times} \mathbf{left}(T)) \wedge \mathrm{D}(\mathbf{right}(S) \vec{\times} T) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(\mathbf{left}(T)) \wedge (\mathrm{C}(S) \vee \mathrm{K}(\mathbf{left}(T))) \\
& \quad \wedge \mathrm{D}(\mathbf{right}(S)) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(\mathbf{right}(S)) \vee \mathrm{K}(T)) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(T)
\end{aligned}$$

How do we represent functions?

Definition (transforms (indexed functions))

A set of transforms $(S, L, \rhd)$ is made up of non-empty sets S and L, and a function

$$\rhd \in L \to (S \to S).$$

We normally write $l \rhd s$ rather than $\rhd(l)(s)$. We can think of $l \in L$ as the index for a function $f_l(s) = l \rhd s$, so $(S, L, \rhd)$ represents the set of function $F = \{f_l \mid l \in L\}$.

Example 3 : mildly abstract description of BGP's ASPATHs

Let $\mathrm{apaths}(X) = (\mathcal{E}(\Sigma^*) \cup \{\infty\},\ \Sigma \times \Sigma,\ \rhd)$ where

$$\begin{array}{rcl} \mathcal{E}(\Sigma^*) & = & \text{finite, elementary sequences over } \Sigma \text{ (no repeats)} \\ (m,\ n) \rhd \infty & = & \infty \\ (m,\ n) \rhd l & = & \begin{cases} n \cdot l & (\text{if } m \notin n \cdot l) \\ \infty & (\text{otherwise}) \end{cases} \end{array}$$

Minimal Sets

Definition (Min-sets)

Suppose that $(S, \lesssim)$ is a pre-ordered set. Let $A \subseteq S$ be finite. Define

$$\begin{aligned} \min_{\lesssim}(A) &\equiv \{a \in A \mid \forall b \in A : \neg(b < a)\} \\ \mathcal{P}(S, \lesssim) &\equiv \{A \subseteq S \mid A \text{ is finite and } \min_{\lesssim}(A) = A\} \end{aligned}$$

Definition (Min-Set Semigroup)

Suppose that $(S, \lesssim)$ is a pre-ordered set. Then

$$\mathcal{P}^{\cup}_{\min}(S, \lesssim) = (\mathcal{P}(S, \lesssim), \oplus^{\lesssim}_{\min})$$

is the semigroup where

$$A \oplus^{\lesssim}_{\min} B \equiv \min_{\lesssim}(A \cup B).$$

Min-Set-Map construction

Definition

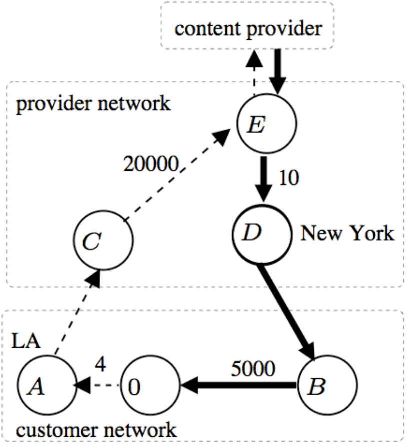
Suppose that $S = (S, \lesssim, F)$ a routing algebra in the style of Sobrinho [Sob03, Sob05]. Then

$$\mathrm{minsetmap}(S) \equiv (\mathcal{P}(S, \lesssim), \oplus^{\lesssim}_{\mathrm{min}}, F^{\lesssim}_{\mathrm{min}})$$

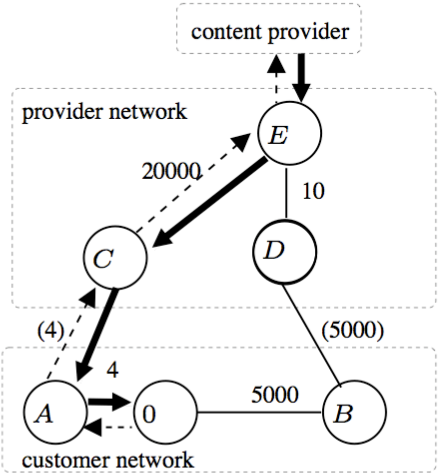
where $F^{\lesssim}_{\mathrm{min}} = \{g_f \mid f \in F\}$ and

$$g_f(A) \equiv \mathrm{min}_{\lesssim}(\{f(a) \mid a \in A\}).$$

Let's turn to BGP MED's — First, hot potato

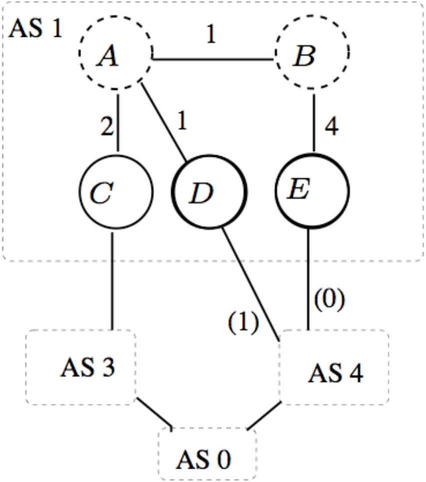


Cold Potato



The (4) represents a MED value.

The System MED-EVIL [MGWR02, Sys].



The values (0) and (1) represent MED values sent by AS 4. The other values are IGP link weights.

Best route selection at nodes A and B.

- r_C, r_D and r_E denote routes received from routers C, D, and E, respectively
- A receives route r_E through route reflector B
- B receives routes r_C and r_D through route reflector A

u	S	BGP best of S at u	due to
A	$\{r_C, r_D\}$	r_D	IGP
A	$\{r_D, r_E\}$	r_E	MED
A	$\{r_E, r_C\}$	r_C	IGP
A	$\{r_C, r_D, r_E\}$	r_C	MED, IGP
B	$\{r_D, r_E\}$	r_E	MED
B	$\{r_E, r_C\}$	r_C	IGP

There is not stable routing!

Assume A always has routes r_C and r_D, so only two cases:

- A knows the routes $\{r_C,\ r_D,\ r_E\}$ and so selects r_C. This implies that B has chosen r_E, and this is a contradiction, since B would have $\{r_E,\ r_C\}$ and select r_C.
- A has only $\{r_C,\ r_D\}$ and selects r_D. Since A does not learn a route from B, we know that B must have selected r_C. This is a contradiction since B would learn r_D from A and then pick r_E.

What's going on with MED?

- Assume MEDs are represented by pairs of the form $(a, \ m)$, where a is an ASN and m is an integer metric.
- The partial order on MEDs is defined as

$$(\alpha_1, \ m) \lesssim_M (\alpha_2, \ n) \equiv \alpha_1 = \alpha_2 \wedge m \lesssim n.$$

- We can think abstractly of BGP routes as elements of

$$(P, \ \lesssim_P) \vec{\times} (M, \ \lesssim_M) \vec{\times} (S, \ \lesssim_S),$$

where $(P, \ \lesssim_P)$ represents the *prefix* of attributes considered before MED, and $(S, \ \lesssim_S)$ represents the *suffix* of attributes considered after MED.

What is going on?

Suppose that we have the lexicographic product,

$$(A, \lesssim_A) \vec{\times} (B, \lesssim_B) \equiv (A \times B, \lesssim),$$

and that W is a finite subset of $A \times B$. We would like to explore efficient (and correct) methods for computing the min-set $\min_{\lesssim}(W)$.

- Let $\sim_A$ and $\sim_B$ be the preorders on A and B for which all elements are related.

Pipeline method

We say the pipeline method is correct when

$$\min_{\lesssim_A \vec{\times} \lesssim_B}(W) = \min_{\sim_A \vec{\times} \lesssim_B}(\min_{\lesssim_A \vec{\times} \sim_B}(W)).$$

Pipeline

Claim

The pipeline method is correct if and only if no two elements of B are strictly ordered, or no two elements of A are incomparable.

Proof : For the the interesting direction, suppose that A does contain two elements a_1 and a_2 with $a_1 \sharp a_2$, and B does contain two elements b_1 and b_2 with $b_1 <_B b_2$. Then

$$\min_{\lesssim_A \vec{\times} \lesssim_B} \{(a_1, b_1), (a_2, b_2)\} = \{(a_1, b_1), (a_2, b_2)\}$$

but

$$\begin{aligned} & \min_{\omega_A \times \lesssim_B} \big(\min_{\lesssim_A \times \omega_B} \{(a_1, b_1), (a_2, b_2)\} \big) \\ = & \min_{\omega_A \times \lesssim_B} \{(a_1, b_1), (a_2, b_2)\} \\ = & \{(a_1, b_1)\}. \end{aligned}$$

So the pipelined decision process does work when we are dealing

Can we generalize the min-set constructions?

Pathfinding through Congruences
Alexander J. T. Gurney, Timothy G. Griffin
12th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 12)
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Semigroup congruence

An equivalence relation $\sim$ on semigroup $(S, \oplus)$ is a congruence if

$$a \sim b \implies (a \oplus c) \sim (b \oplus c) \wedge (c \oplus a) \sim (c \oplus b)$$

$(S/\sim, \oplus_\sim)$ is a semigroup

$$[a] \oplus_\sim [b] = [a \oplus b]$$

Reductions [Won79]

If $(S, \oplus)$ is a semigroup and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$

For monoids the first axioms is not needed since $r(a \oplus 0) = r(r(a) \oplus 0)$ from the second axiom.
Similarly, the second axiom can be simplified to a single equality in the case of a commutative semigroup.

Reductions on Semirings

A function on a semiring is called a reduction if it is a reduction with respect to both of the semiring operations.
Similarly, a reduction on a semigroup transform $(S, \oplus, F)$ is a function r from S to itself, such that r is a reduction on $(S, \oplus)$ and

$$r(f(a)) = r(f(r(a))) \tag{1}$$

for all a in S and f in F.

Lemma

For any reduction r on $(S, \oplus)$*, define a relation* $\sim_r$ *on S by*

$$a \sim_r b \overset{\text{def}}{\iff} r(a) = r(b).$$

This $\sim_r$ *is a congruence.*

Proof.

This is obviously an equivalence relation. To prove that it is a congruence, suppose that $a \sim_r b$, so that $r(a) = r(b)$. Then

$$r(a \oplus c) = r(r(a) \oplus c) = r(r(b) \oplus c) = r(b \oplus c)$$

and likewise for $r(c \oplus a) = r(c \oplus b)$. Hence $\sim_r$ is indeed a congruence. □ □

Lemma

Let $(S, \oplus)$ *be a semigroup,* $\sim$ *a congruence, and* $\rho^\natural$ *the natural map. If* $\theta : S/\sim \longrightarrow S$ *is such that* $\rho^\natural \circ \theta = id$*, then* $\theta \circ \rho^\natural$ *is a reduction; and* $\sim$ *is equal to* $\sim_{\theta \circ \rho^\natural}$.

- We can represent any reduction *r* as a pair $(\sim, \theta)$

Specifically, for a given $(S, \oplus, F)$ and reduction $r : S \longrightarrow S$ we can define the quotient S/r as follows.

1. The carrier consists of r-equivalence classes of elements of S; we can choose the canonical representative of each class to be a fixed point of r.
2. The semigroup operation is given by $\rho^\natural(a) \oplus/r\ \rho^\natural(b) = \rho^\natural(a \oplus b)$.
3. The functions in F are lifted: $f(\rho^\natural(a)) = \rho^\natural(f(a))$.

This can be verified to be a semigroup transform. The minset construction is clearly a special case, where r is min, S is a set of sets, and $\oplus$ is set union.

Modeling Path Errors?

- The same node is visited more than once.
- The path is intended to be filtered out.
- The path violates known economic relationships between networks.
- The path is too long (exceeding a maximum size for routing announcements).
- The origin is unexpected (given neighbours are only anticipated to advertise certain addresses).
- Route data is otherwise malformed.

Only Simple Paths

$S \vec{\times} P$

- $(S, \leq, F)$ be an order transform for encoding the path weights.
- P be the algebra of paths $(N^*, \preceq, C)$, where $p \preceq q$ if and only if $\mid p \mid \leq \mid q \mid$, and C consists of functions c_n for all n in N, which concatenate the node n onto the given path.

Bad paths $B \subseteq S \times N^*$

$$B \equiv \{(s, p) \in S \times N^* \mid p \text{ is not simple}\}.$$

A reduction over subsets of $S \times N^*$

$$r(A) \stackrel{\text{def}}{=} \min(A \setminus E); \tag{2}$$

where min uses the lexicographic order on $S \times N^*$.

The construction...

A semigroup transform can be constructed where

- the elements are those subsets of $S \times N^*$ which are fixed points of r;
- the operation $\oplus$ is given by $A \oplus B \stackrel{\text{def}}{=} r(A \cup B)$; and
- the functions are pairs (f, c_n) for f in F, where

$$(f, c_n)(A) \stackrel{\text{def}}{=} r(\{(f(s), c_n(p)) \mid (s, p) \in A\}).$$

It can be seen that this algebra implements the simple paths criterion in the case of multipath routing: if during the course of computation a non-simple path is computed, it and its associated S-value will be removed from the candidate set.

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